Bifurcation and Chaos in Non-smooth Mechanical Systems

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ANALYSIS, CONTROL, SIMULATION AND APPLICATIONS"

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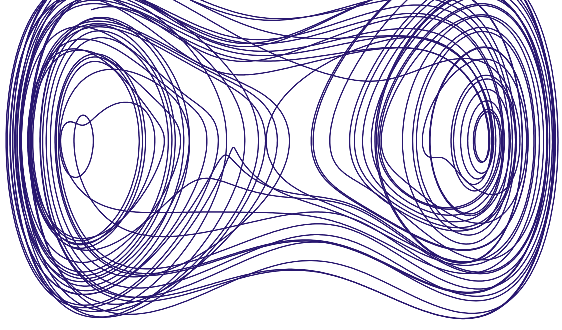


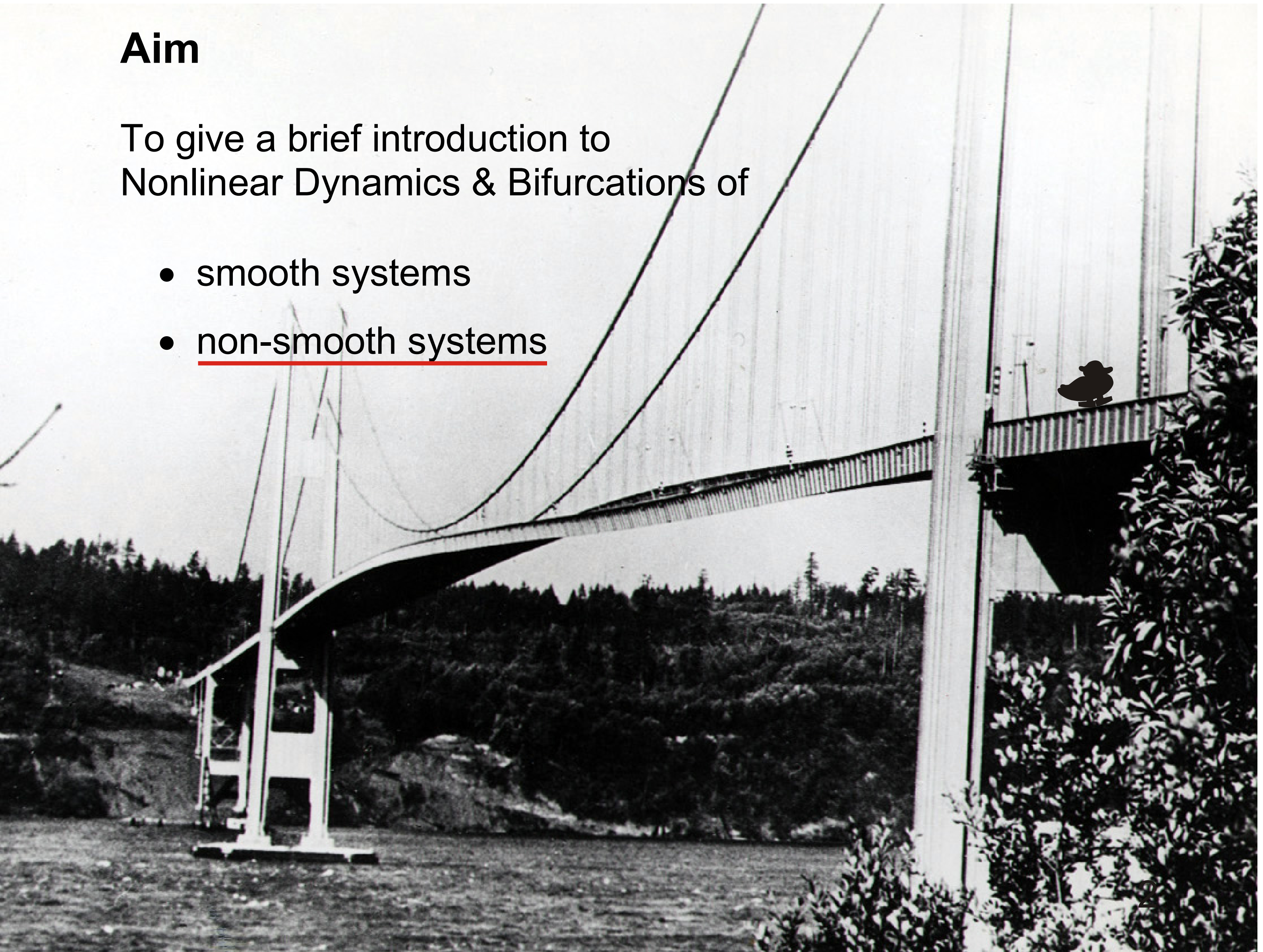
1 Introduction

Contents

Aim

To give a brief introduction to
Nonlinear Dynamics & Bifurcations of

- smooth systems
- non-smooth systems



2 Non-smooth Dynamical Systems

A dynamical system is a system whose state evolves with time. The evolution is governed by a set of rules (usually differential equations).

A dynamical system can be non-smooth...

2.1 Continuous-time Dynamical Systems

smooth system

$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t))$ rhs is differentiable up to any order

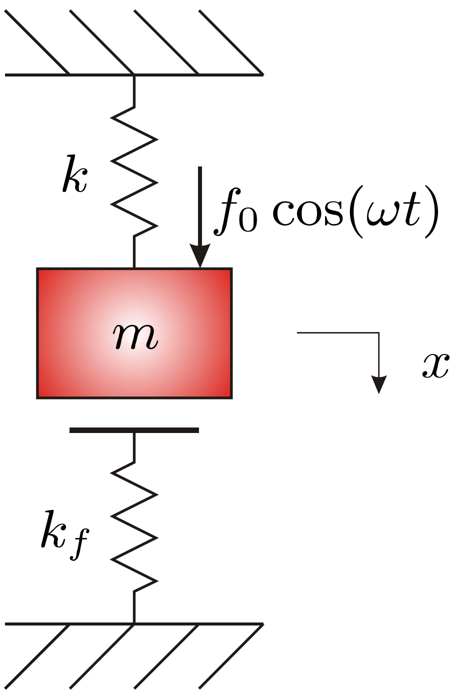
non-smooth continuous system

$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t))$ rhs is continuous but non-smooth

Example:

$$m\ddot{x} + kx = f_0 \cos(\omega t) - f(x)$$

$$f(x) = \begin{cases} 0, & x \leq 0 \\ k_f x, & x > 0 \end{cases}$$



Filippov system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t)) = \begin{cases} \boldsymbol{f}_-(t, \boldsymbol{x}(t)), & \boldsymbol{x} \in \mathcal{V}_- \\ \boldsymbol{f}_+(t, \boldsymbol{x}(t)), & \boldsymbol{x} \in \mathcal{V}_+ \end{cases}$$

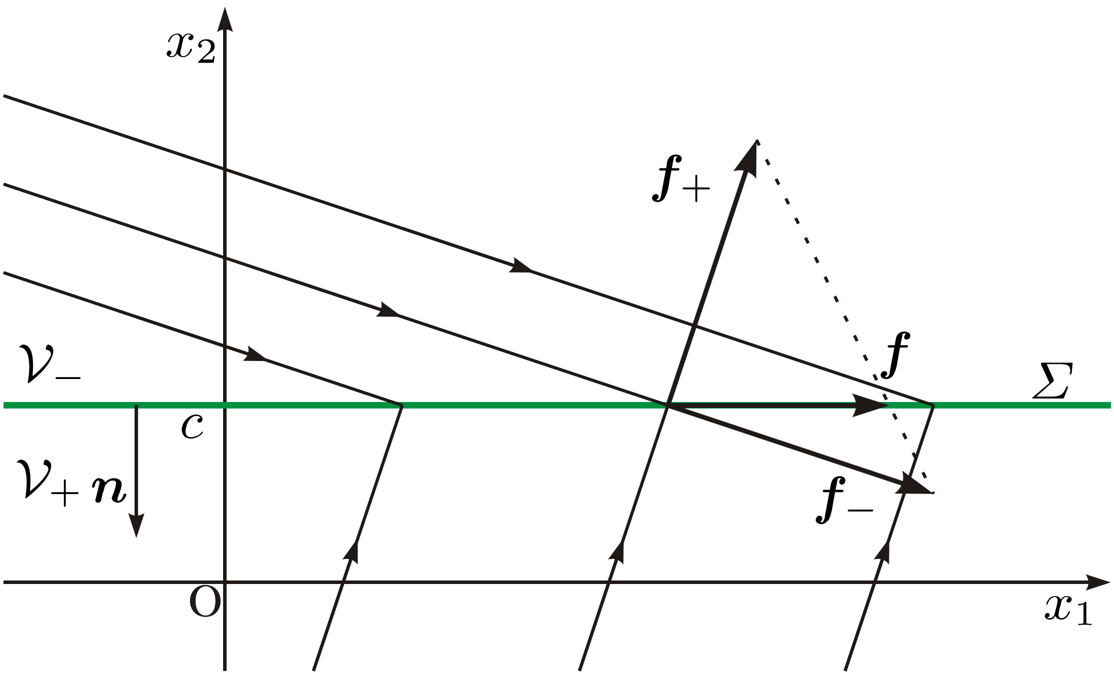
not everywhere

rhs is discontinuous on hypersurfaces Σ

replace with differential inclusion

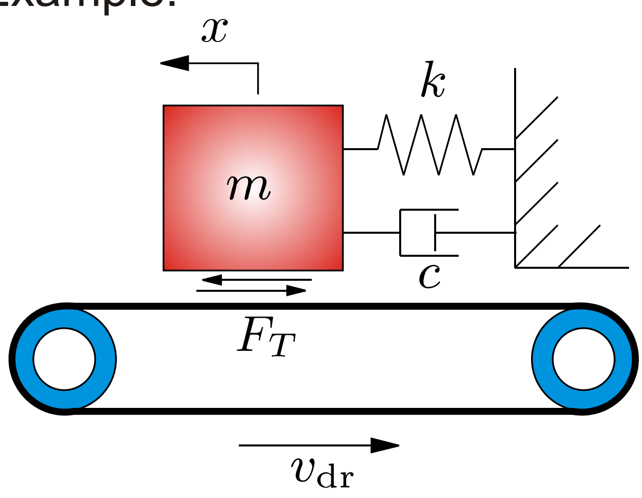
$$\dot{\boldsymbol{x}}(t) \in \boldsymbol{F}(t, \boldsymbol{x}(t)) = \begin{cases} \boldsymbol{f}_-(t, \boldsymbol{x}(t)), & \boldsymbol{x} \in \mathcal{V}_- \\ \overline{\text{co}}\{\boldsymbol{f}_-(t, \boldsymbol{x}(t)), \boldsymbol{f}_+(t, \boldsymbol{x}(t))\}, & \boldsymbol{x} \in \Sigma \\ \boldsymbol{f}_+(t, \boldsymbol{x}(t)), & \boldsymbol{x} \in \mathcal{V}_+ \end{cases}$$

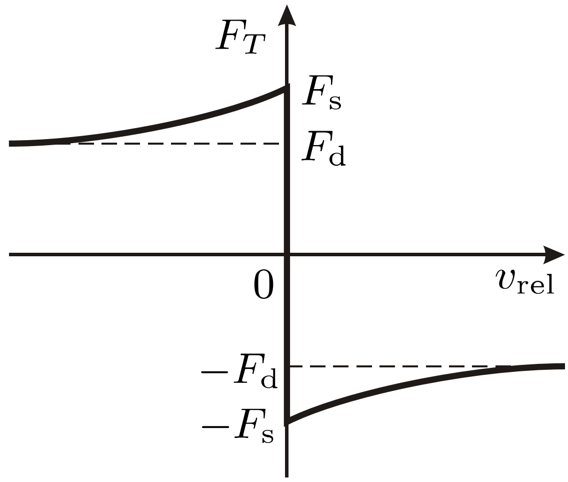
almost everywhere

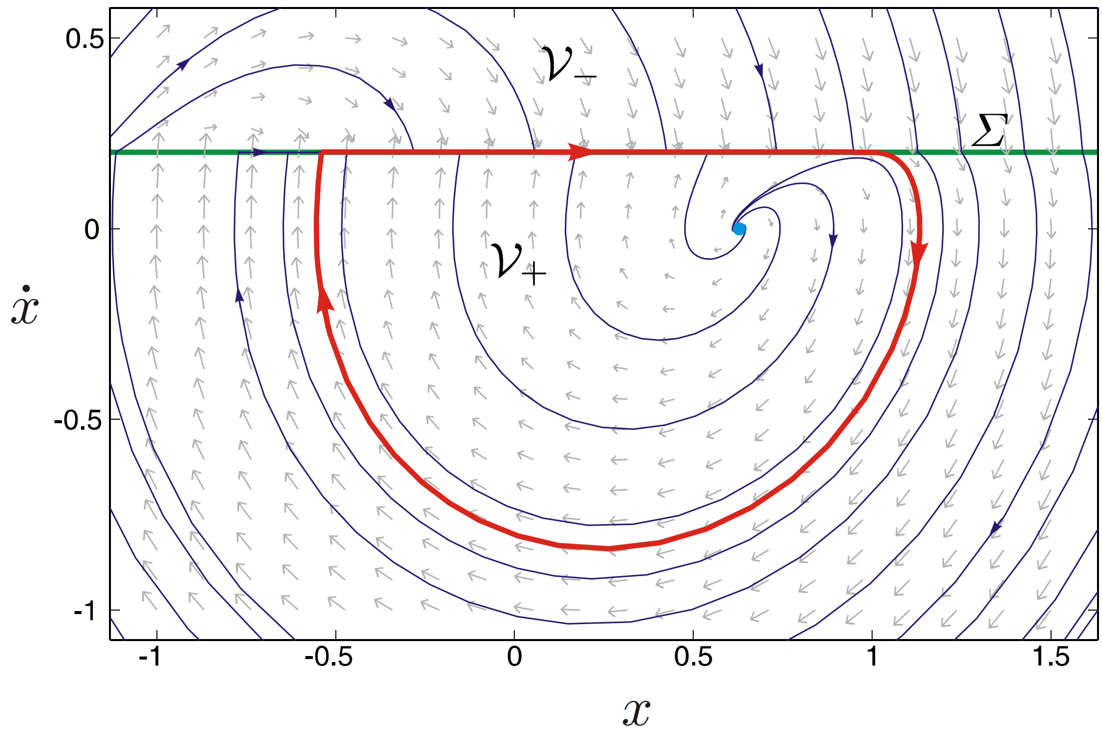


attractive sliding mode

Example:







measure differential inclusion

$$\mathrm{d}\boldsymbol{x} \in \boldsymbol{F}(t, \boldsymbol{x}(t))\mathrm{d}t + \boldsymbol{G}(t, \boldsymbol{x}(t))\mathrm{d}\eta \qquad \text{or} \qquad \mathrm{d}\boldsymbol{x} \in \mathrm{d}\boldsymbol{\Gamma}(t, \boldsymbol{x}(t))$$

with $\mathrm{d}\boldsymbol{x} = \dot{\boldsymbol{x}}(t)\mathrm{d}t + (\boldsymbol{x}^+(t) - \boldsymbol{x}^-(t))\mathrm{d}\eta$

A measure differential inclusion is able to describe discontinuities in the state.

2.2 Discrete-time Dynamical Systems

A discrete-time dynamical system is described by a mapping $\boldsymbol{P} : \mathbb{R}^n \to \mathbb{R}^n$

$$\boldsymbol{y}_{i+1} = \boldsymbol{P}(\boldsymbol{y}_i)$$

The iteration parameter i defines a "discrete time".

The mapping can be iterated:

$$\boldsymbol{y}_{i+2} = \boldsymbol{P}(\boldsymbol{y}_{i+1}) = \boldsymbol{P}(\boldsymbol{P}(\boldsymbol{y}_i)) =: \boldsymbol{P}^2(\boldsymbol{y}_i)$$

Example: Predator-Prey Model

x_i number of rabbits on Malta in year i

y_i number of foxes on Malta in year i

reproduction

$$x_{i+1} = x_i(a - \alpha y_i)$$

predation

$$y_{i+1} = y_i(-b + \beta x_i)$$

starvation

3 Steady-state Behaviour

A dynamical system expressed by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}) \quad \text{or} \quad \dot{\boldsymbol{x}} \in \boldsymbol{F}(t, \boldsymbol{x}) \quad \text{or} \quad \mathrm{d}\boldsymbol{x} \in \mathrm{d}\boldsymbol{\Gamma}(t, \boldsymbol{x})$$

with initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ defines a solution curve or trajectory in the state-space which we denote by $\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0)$ or simply by $\boldsymbol{x}(t)$.

A solution of an initial value problem of a non-smooth system is not always unique and might even not exist!

Some special solutions

equilibria, (quasi)-periodic solutions and chaotic solutions

are called steady-states of the system.

A limit point/set is a point or set of points in the state-space which can be approached in forward or backward time.

attracting, repelling or saddle-type of steady-states are limit points/sets.

3.1 Equilibria

An equilibrium $\boldsymbol{x}^*$ is a point for which there exists a solution curve $\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}^*) = \boldsymbol{x}^*$ for all $t \geq t_0$

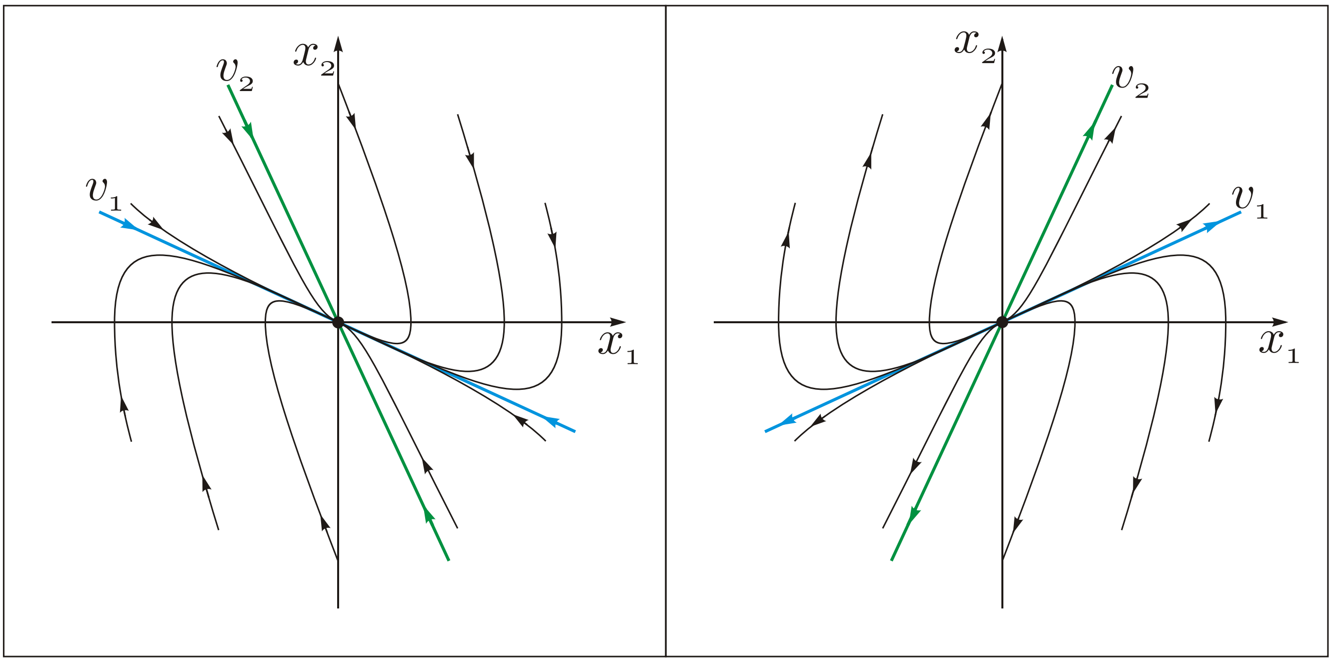
i.e. an equilibrium is a constant solution of the system and it holds that

$$\mathbf{0} = \boldsymbol{f}(t, \boldsymbol{x}^*) \quad \text{or} \quad \mathbf{0} \in \boldsymbol{F}(t, \boldsymbol{x}^*) \quad \text{or} \quad \mathbf{0} \in \mathrm{d}\boldsymbol{\Gamma}(t, \boldsymbol{x}^*)$$

Types of equilibria of a linear planar system

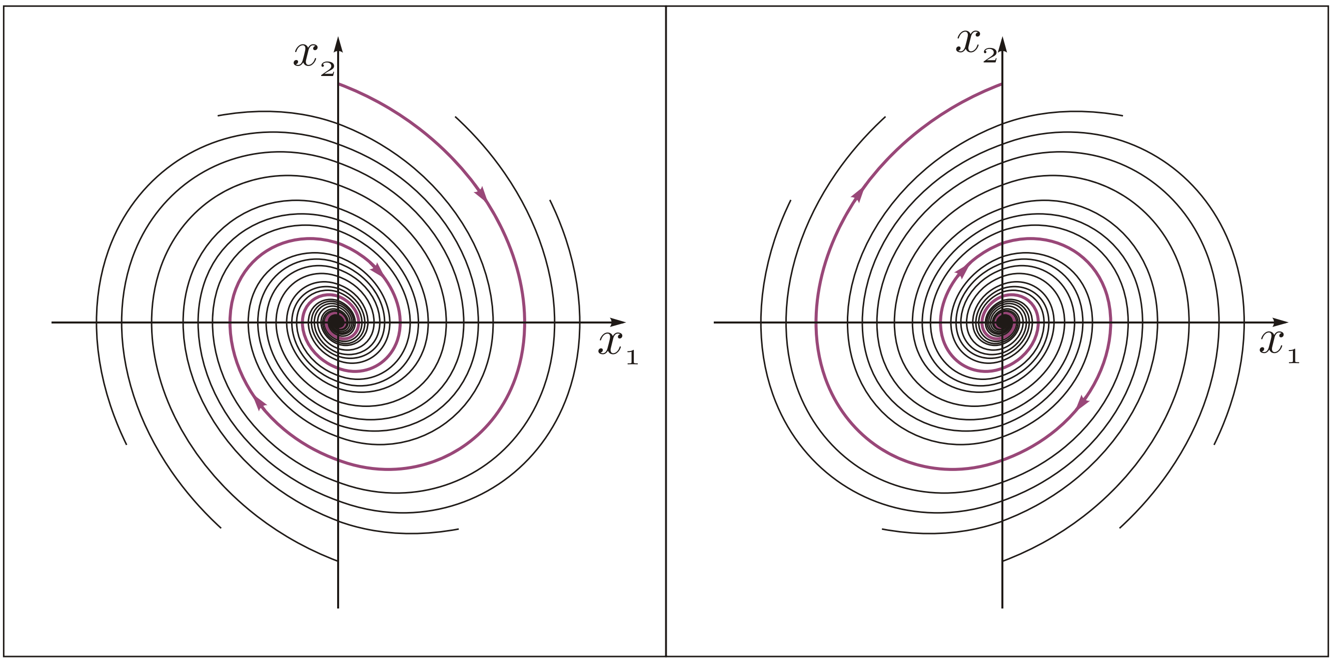
$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x}, \quad \boldsymbol{x} \in \mathbb{R}^2, \quad (\lambda_1, \lambda_2) = \mathrm{eig}(\boldsymbol{A})$$

Node $\lambda_1, \lambda_2 < 0$ or $\lambda_1, \lambda_2 > 0$ real



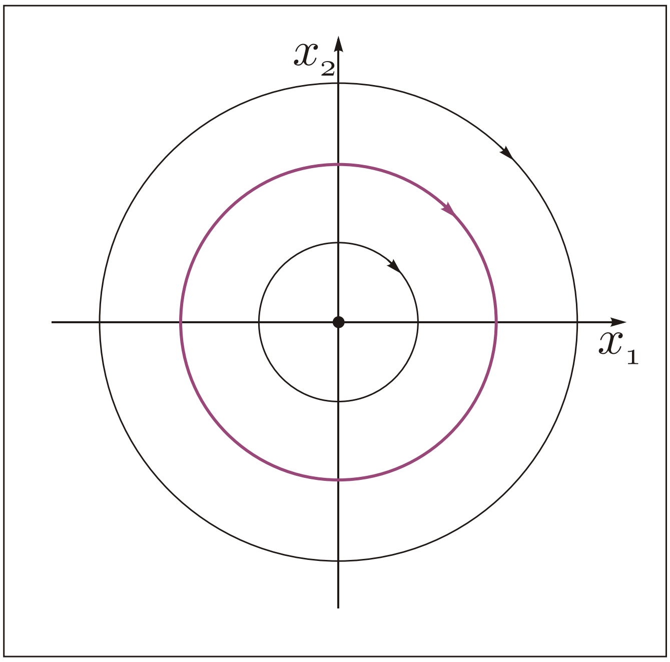
stable $\lambda_1 < 0,\ \lambda_2 < 0$ unstable $\lambda_1 > 0,\ \lambda_2 > 0$

Focus $\lambda_1 = \overline{\lambda}_2$, $\mathrm{Im}(\lambda_{1,2}) \neq 0$, $\mathrm{Re}(\lambda_{1,2}) \neq 0$



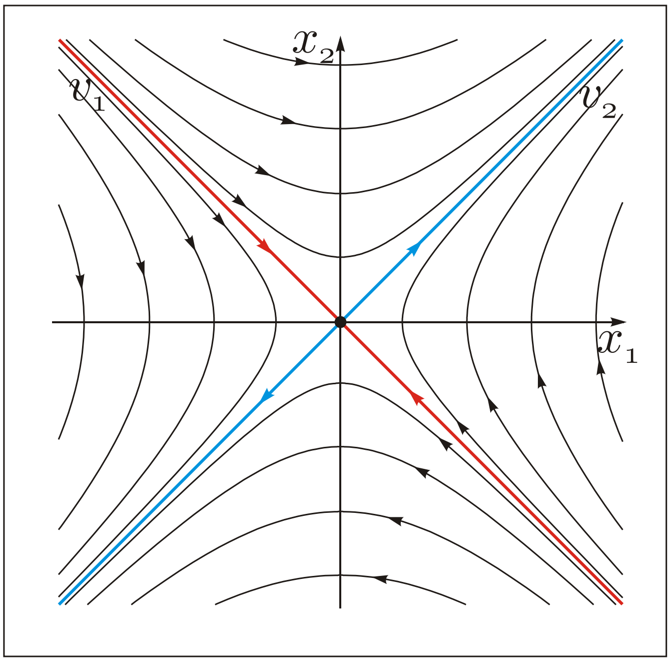
stable $\mathrm{Re}(\lambda_{1,2}) < 0$ unstable $\mathrm{Re}(\lambda_{1,2}) > 0$

Center $\lambda_1 = \overline{\lambda}_2,\ \mathrm{Im}(\lambda_{1,2}) \neq 0,\ \mathrm{Re}(\lambda_{1,2}) = 0$



not a limit point!

Saddle λ_1, λ_2 real, $\lambda_1 < 0,\ \lambda_2 > 0$ (or $\lambda_1 > 0,\ \lambda_2 < 0$)



3.2 Periodic Solutions

Non-autonomous systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x})$

A trajectory for which holds

$$\boldsymbol{\varphi}(t, t_0, \boldsymbol{x}_0) = \boldsymbol{\varphi}(t + T, t_0, \boldsymbol{x}_0)$$

is called a periodic solution. $(t_0, \boldsymbol{x}_0)$ is a point on the periodic solution. T is the period time and is the minimal period for which the periodicity property holds.

Differentiation: $\dot{\boldsymbol{\varphi}}(t, t_0, \boldsymbol{x}_0) = \dot{\boldsymbol{\varphi}}(t + T, t_0, \boldsymbol{x}_0)$

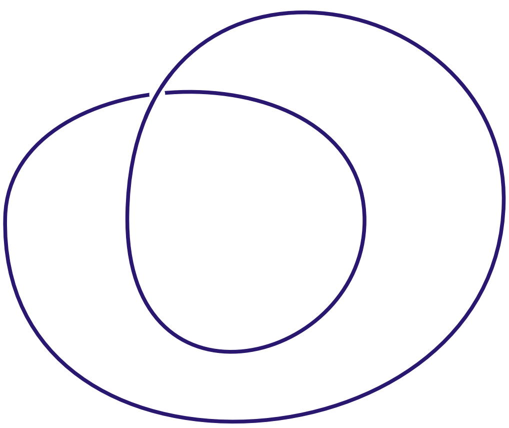
$$\Rightarrow \boldsymbol{f}(t, \boldsymbol{x}(t)) = \boldsymbol{f}(t + T, \boldsymbol{x}(t))$$

System is also periodic!

Generally $\boldsymbol{f}(t, \boldsymbol{x}(t)) = \boldsymbol{f}(t + \tau, \boldsymbol{x}(t))$

with $k\tau = T, \qquad k = 1, 2, 3, \ldots$

A periodic solution with $T = k\tau$ is a called a period-k solution.



Autonomous systems $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$

A trajectory for which holds

$$\boldsymbol{\varphi}(t, t_a, \boldsymbol{x}_0) = \boldsymbol{\varphi}(t + T, t_a, \boldsymbol{x}_0) \qquad \forall t_a \; \forall t$$

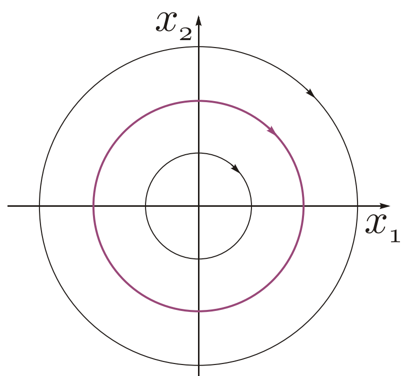
is called a periodic solution. $\boldsymbol{x}_0$ is a point on the periodic solution. T is the period time and is the minimal period for which the periodicity property holds.

A periodic solution of an autonomous system can be k -periodic with respect to a Poincaré map (as we will see..).

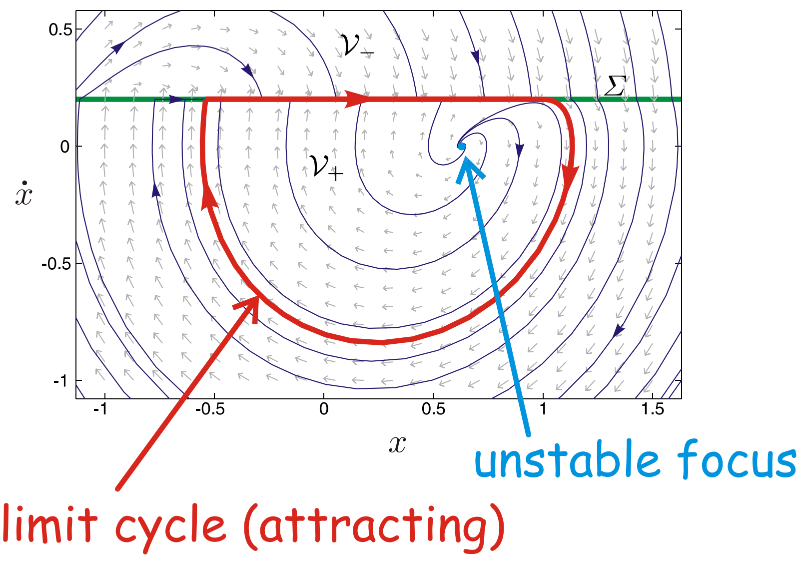
An isolated periodic solution of a system (autonomous or non-autonomous), which is a limit set, is called a limit cycle.

Example:

Center



infinitely many periodic solutions



3.3 Quasi-Periodic Solutions

Two frequencies ω_1 and ω_2 are incommensurate if ω_1/ω_2 is an irrational number.

A quasi-periodic solution is characterised by two or more incommensurate frequencies.

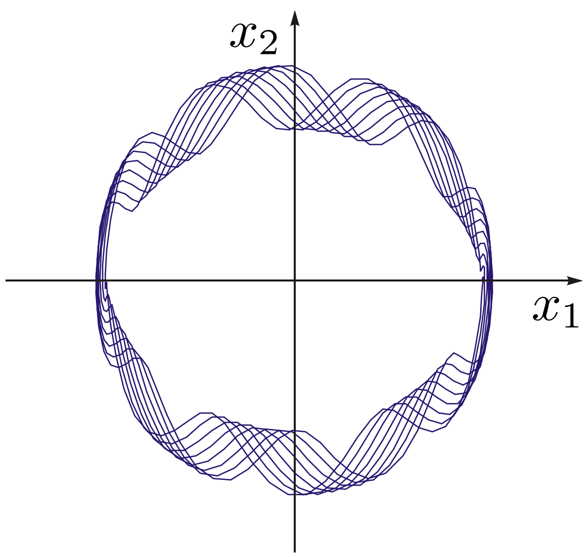
Example:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\omega^2 x_1 + F\cos(\Omega t)\end{aligned} \qquad \Omega \neq \omega \quad \text{no resonance}$$

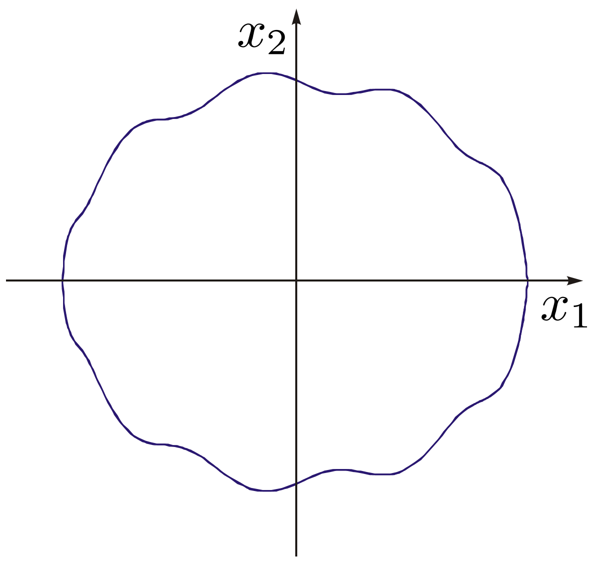
$$\left.\begin{aligned} x_1(t) &= a\cos(\omega t) + b\sin(\omega t) + \tfrac{F}{\omega^2-\Omega^2}\cos(\Omega t) \\ x_2(t) &= -a\omega\sin(\omega t) + b\omega\cos(\omega t) - \tfrac{F\Omega}{\omega^2-\Omega^2}\cos(\Omega t)\end{aligned}\right\} (*)$$

$\omega = 2, \quad \Omega = 10\sqrt{2} \quad \rightarrow (*)$ is quasi-periodic

$\omega = \pi, \quad \Omega = 10\pi \quad \rightarrow (*)$ is periodic with $T = \frac{2\pi}{\pi} = 2$ sec



quasi-periodic
dense solution on a torus



periodic

3.4 Chaos

not very constructive definition

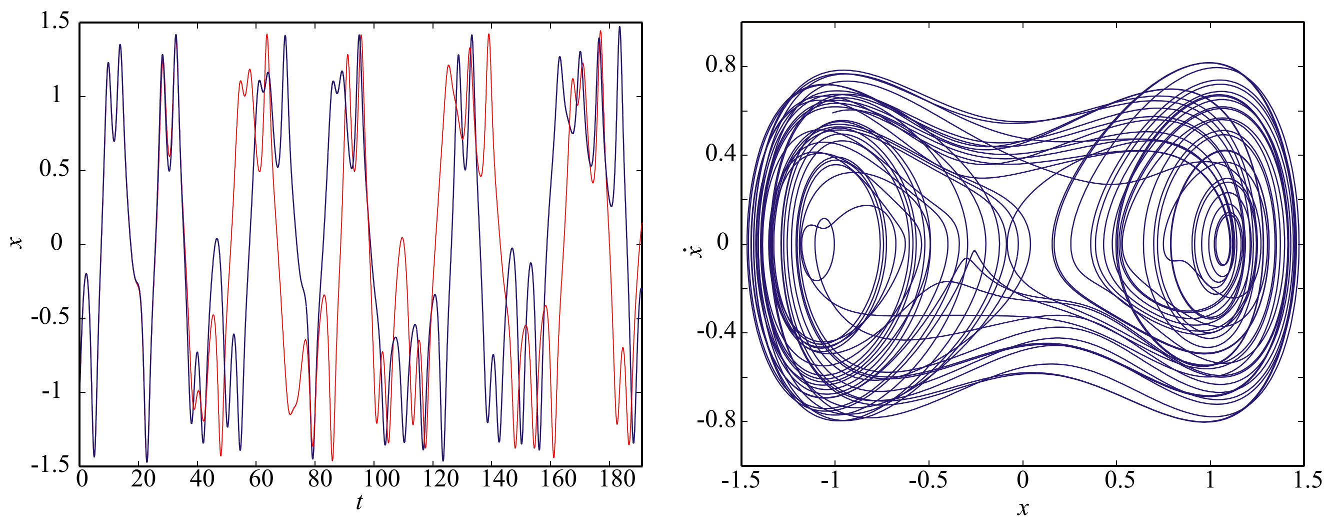
smooth dynamical systems:

Chaos in a deterministic dynamical system is bounded steady-state behaviour that is not a equilibrium, periodic solution or quasi-periodic solution.

A chaotic limit set is, when it is attracting, called a chaotic attractor or strange attractor.

Example: Duffing system $\ddot{x} + \varepsilon \dot{x} - x + x^3 = \gamma \cos(\omega t)$

$$\varepsilon = \frac{1}{4}, \quad \gamma = 0.3, \quad \omega = 1$$



aperiodic oscillation with sensitive dependence on initial conditions; fractal structure

Lyapunov exponents: $e_{Li} = \lim_{t\to\infty} \frac{1}{t} \ln |\lambda_i(t)|$

Floquet multiplier

non-smooth dynamical systems: ??

3.5 Fixed Points

A fixed point $\boldsymbol{y}^*$ of a system $\boldsymbol{y}_{i+1} = \boldsymbol{P}(\boldsymbol{y}_i)$ is a point which is mapped to itself

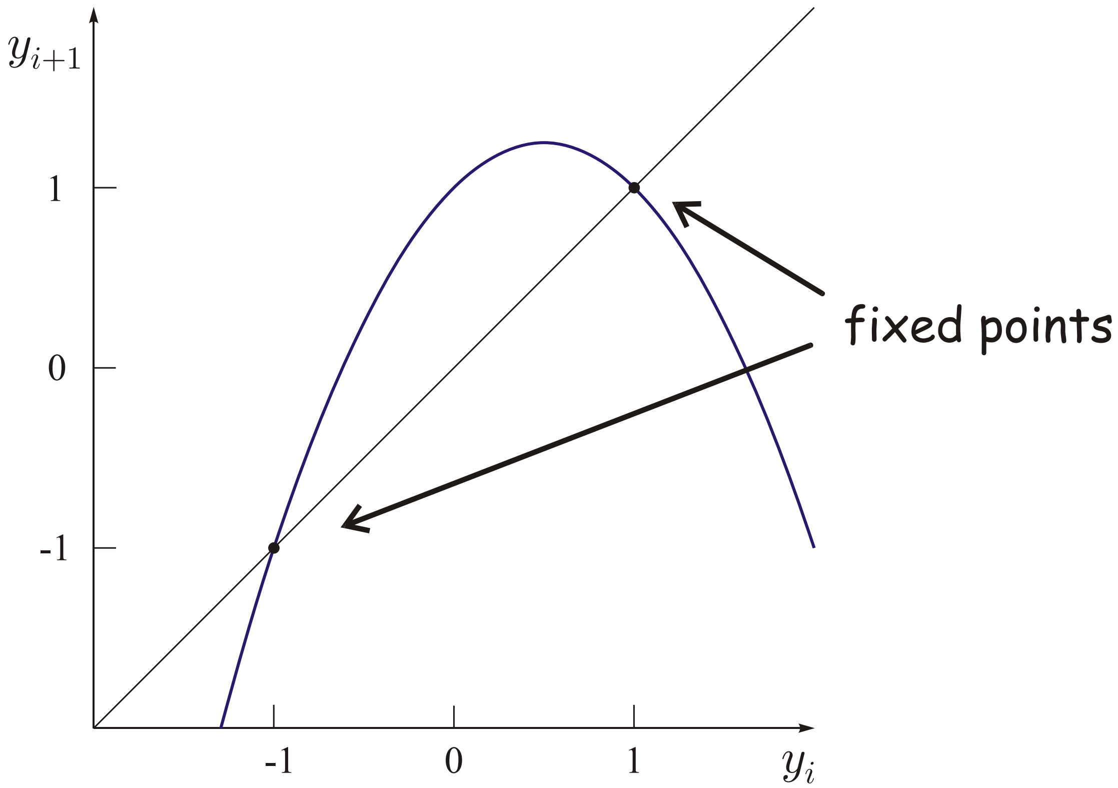
$$\boldsymbol{y}^* = \boldsymbol{P}(\boldsymbol{y}^*)$$

A point $\boldsymbol{y}^*$ which satisfies

$$\boldsymbol{y}^* = \boldsymbol{P}^k(\boldsymbol{y}^*)$$

is called a period k-point and is a fixed point of $\boldsymbol{P}^k(\boldsymbol{y}^*)$.

Quasi-periodic and chaotic solutions are also possible in a discrete-time system.



4 Definitions of Bifurcation

Geometric Definition of Bifurcation

Poincaré (1905)

A bifurcation of a dynamical system

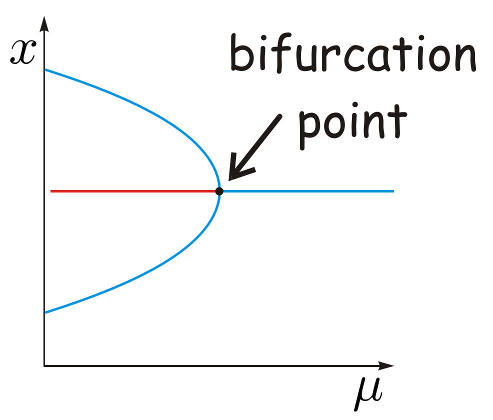
$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu), \ \ \dot{\boldsymbol{x}} \in \boldsymbol{F}(\boldsymbol{x}, \mu), \ \ \mathrm{d}\boldsymbol{x} \in \mathrm{d}\boldsymbol{\Gamma}(\boldsymbol{x}, \mu), \ \boldsymbol{y}_{i+1} = \boldsymbol{P}(\boldsymbol{y}_i, \mu)$$

is a change in the number of steady-states under influence of parameter μ.

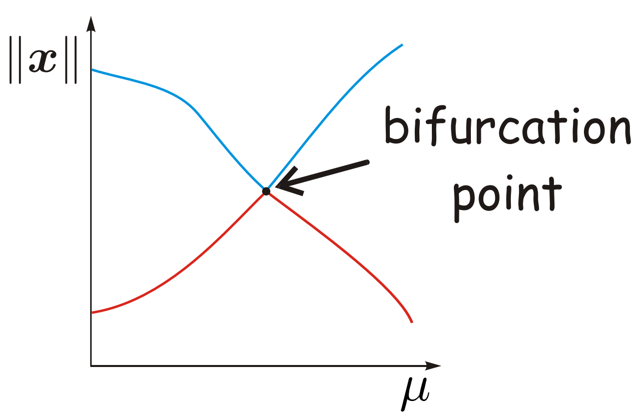
A diagram depicting some scalar measure $[\boldsymbol{x}]$ of $\boldsymbol{x} \in \mathbb{R}^n$ versus μ, where $(\boldsymbol{x}, \mu)$ is on a steady state, is called a bifurcation diagram.

Examples

$$\dot{x} = -\mu x - \alpha x^3$$



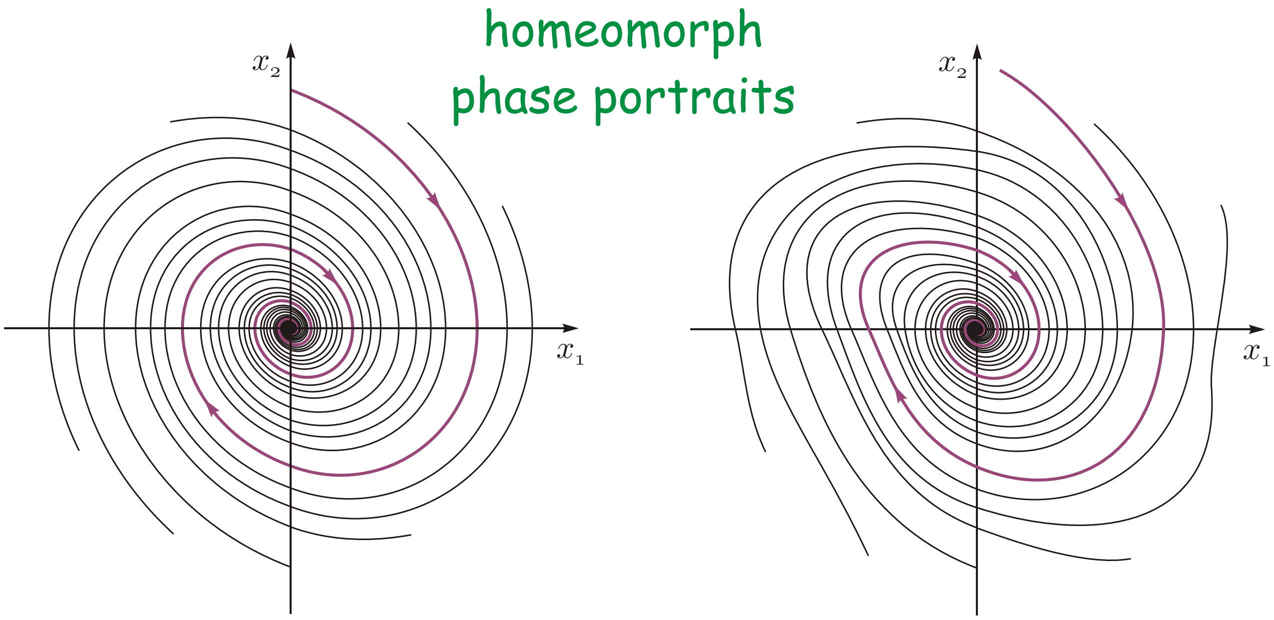
$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu)$$



change: 3-1-1
⇒bifurcation

change: 2-1-2
⇒bifurcation

Homeomorph = identical after stretching and scaling



A homeomorphism is a continuous invertible map.

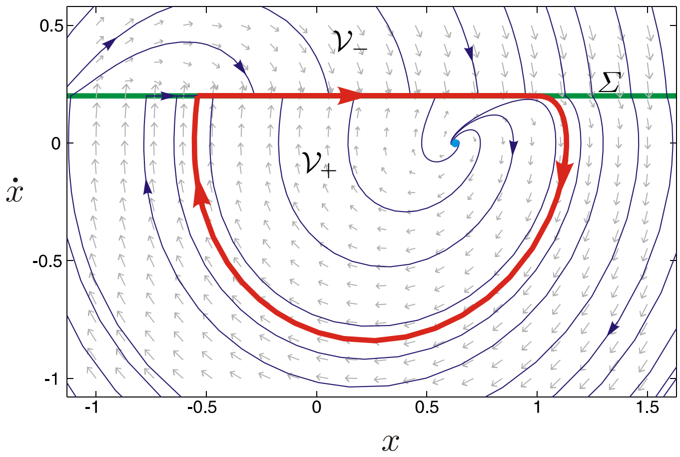
Topological Definition of Bifurcation

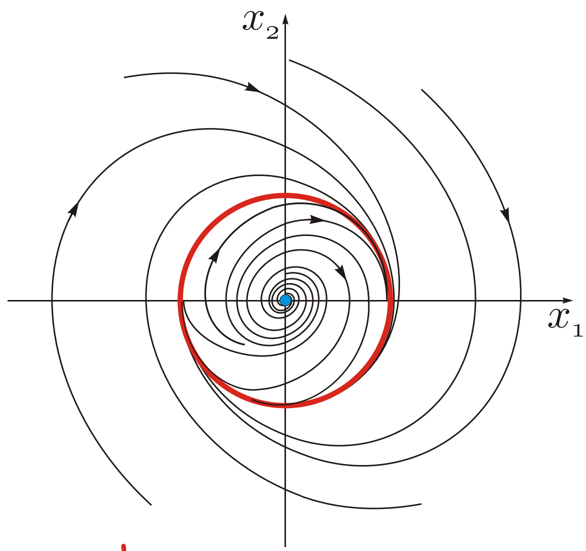
mathematicians

The appearance of topologically nonequivalent phase portraits under variation of a parameter μ is called a bifurcation.

The geometric definition and the topological definition agree for smooth dynamical systems.

phase portrait of a non-smooth dynamical system





not homeomorph
phase portraits!!!

topological definition $\neq$ geometric definition
for non-smooth dynamical systems

In the following, we use the geometric definition of bifurcation.

5 Bifurcations of Equilibria

An equilibrium branch is described by

$$\mathbf{0} = \boldsymbol{f}(\boldsymbol{x}, \mu)$$

Along an equilibrium branch it holds that

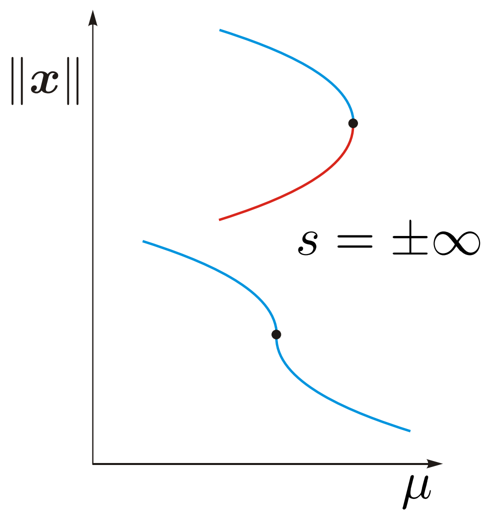
$$\mathbf{0} = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\mu} + \frac{\partial \boldsymbol{f}}{\partial \mu}$$

slope of the branch:

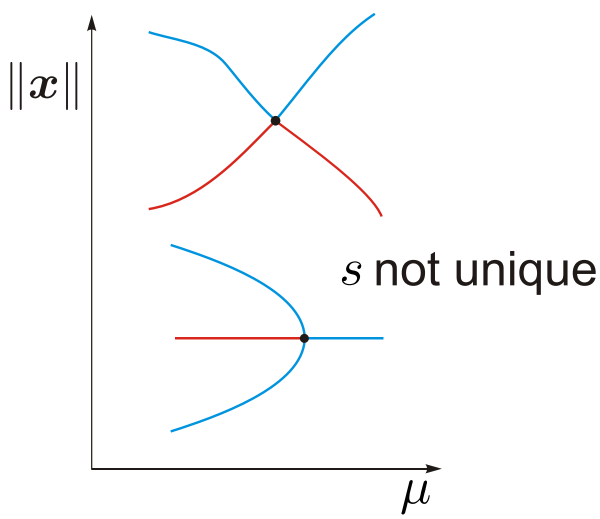
$$s = \frac{\mathrm{d}[\boldsymbol{x}]}{\mathrm{d}\mu} = \frac{\mathrm{d}[\boldsymbol{x}]}{\mathrm{d}\boldsymbol{x}} \frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\mu} = -\frac{\mathrm{d}[\boldsymbol{x}]}{\mathrm{d}\boldsymbol{x}} \left(\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right)^{-1} \frac{\partial \boldsymbol{f}}{\partial \mu}$$

with $\boldsymbol{J}(\boldsymbol{x}, \mu) = \dfrac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}$ regular $\quad \Rightarrow \det(\boldsymbol{J}) \neq 0$

If $\det(\boldsymbol{J}) = 0$, then s is undefined



$\begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} & \frac{\partial \boldsymbol{f}}{\partial \mu} \end{bmatrix}$ full rank

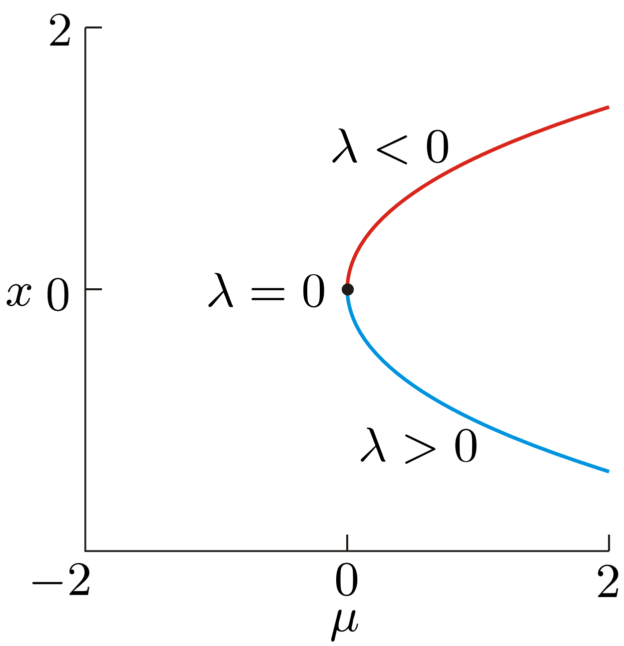


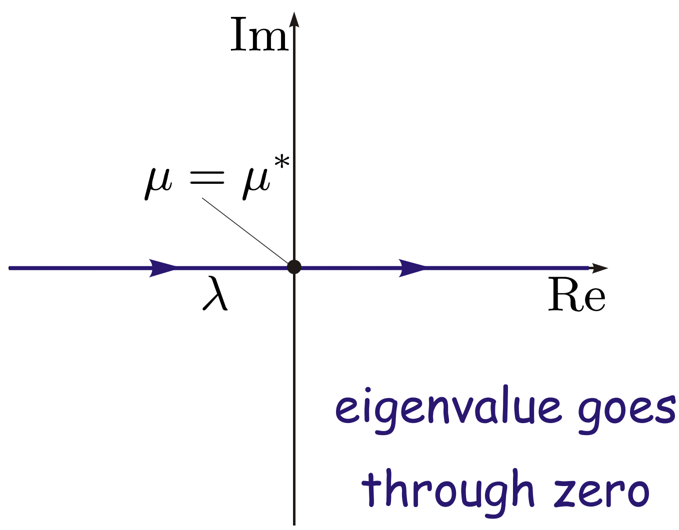
$\begin{bmatrix} \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} & \frac{\partial \boldsymbol{f}}{\partial \mu} \end{bmatrix}$ not full rank

5.1 Saddle-Node Bifurcation (Turning Point Bif.)

normal form: $\dot{x} = f(x, \mu) = \mu - x^2$

equilibria: $x_1^* = \sqrt{\mu}$ $x_2^* = -\sqrt{\mu}$ for $\mu > 0$





Jacobian: $J(x) = \frac{\partial f}{\partial x} = -2x$

eigenvalues: $$\lambda_1 = \left.\frac{\partial f}{\partial x}\right|_{x=x_1^*} = -2\sqrt{\mu}$$

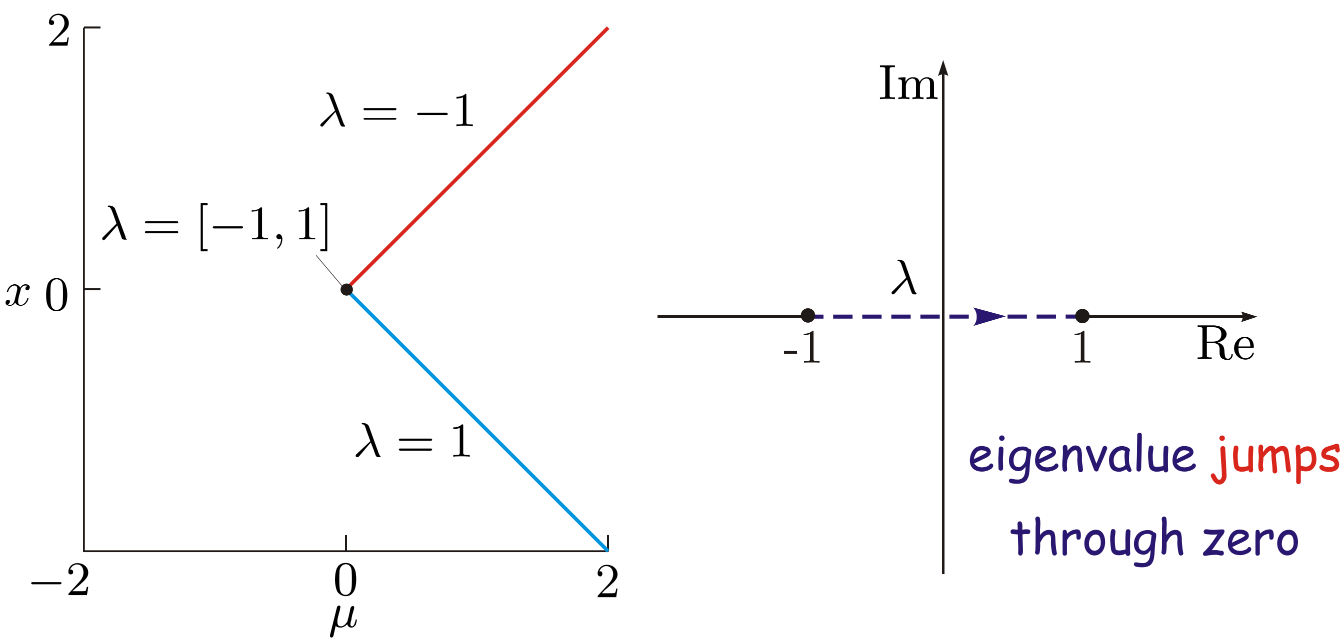
$$\lambda_2 = \left.\frac{\partial f}{\partial x}\right|_{x=x_2^*} = 2\sqrt{\mu}$$

slope at the bifurcation point: $\frac{\partial f}{\partial \mu} = 1 \quad \Rightarrow s = \pm\infty$

non-smooth continuous system:

$$\dot{x} = f(x, \mu) = \mu - |x|$$

equilibria: $x_1^* = \mu \qquad x_2^* = -\mu \qquad$ for $\mu > 0$



discontinuous saddle-node bifurcation

Jacobian: $J(x) = -\operatorname{Sign}(x)$ generalised differential!

$$\operatorname{Sign}(x) = \begin{cases} -1 & x < 0, \\ [-1, 1] & x = 0, \\ 1 & x > 0. \end{cases}$$

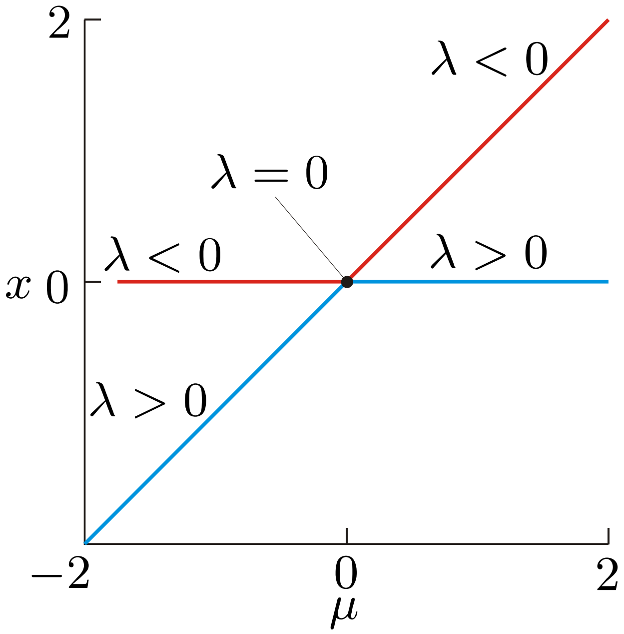
set-valued sign-function

eigenvalue jumps though 0!

or, eigenvalue is set-valued at bif.point with 0 in its set.

5.2 Transcritical Bifurcation

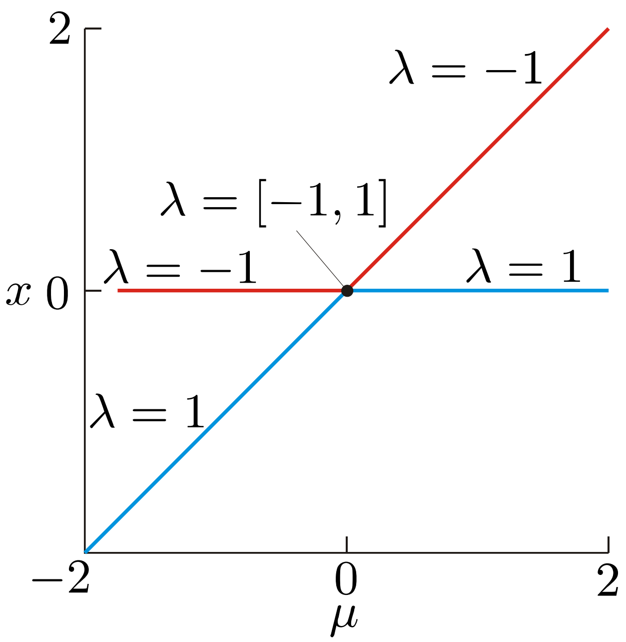
smooth: $\dot{x} = f(x, \mu) = \mu x - x^2$



asymmetric

buckling

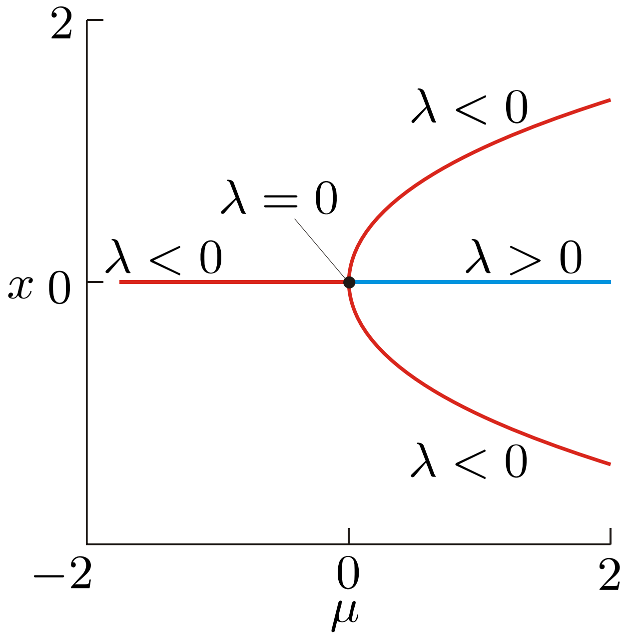
non-smooth: $\dot{x} = f(x, \mu) = |\frac{1}{2}\mu| - |x - \frac{1}{2}\mu|$



discontinuous transcritical bifurcation

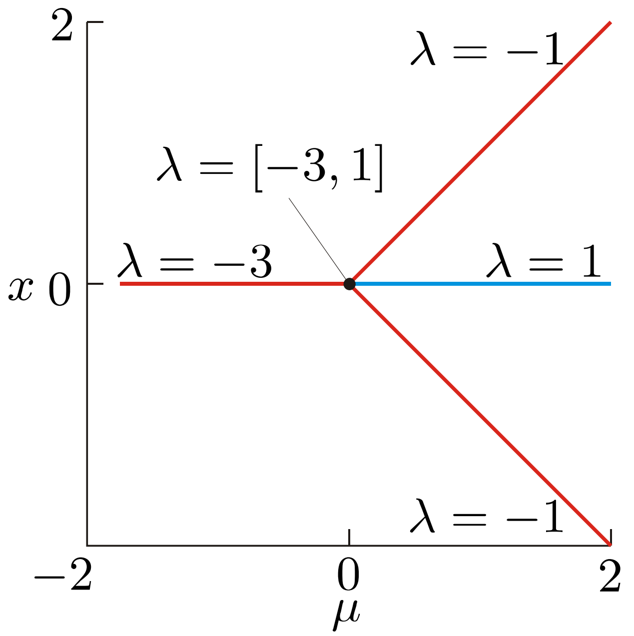
5.3 Pitchfork Bifurcation

smooth: $\dot{x} = f(x, \mu) = \mu x + \alpha x^3$ $\alpha = -1$



symmetric
buckling

non-smooth: $\dot{x} = f(x, \mu) = -x + |x + \frac{1}{2}\mu| - |x - \frac{1}{2}\mu|$



discontinuous pitchfork
bifurcation

5.4 Hopf Bifurcation

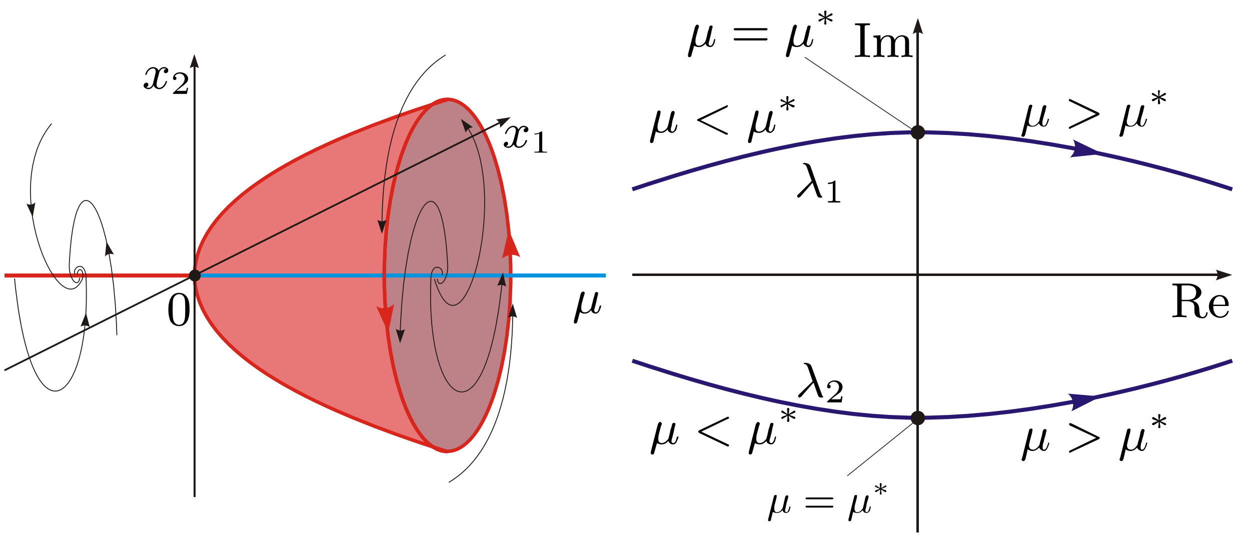
normal form:

$$\dot{x}_1 = \mu x_1 - \omega x_2 + (\alpha x_1 - \beta x_2)(x_1^2 + x_2^2)$$
$$\dot{x}_2 = \omega x_1 + \mu x_2 + (\beta x_1 + \alpha x_2)(x_1^2 + x_2^2)$$

$$\alpha = -1$$

Jacobian at equilibrium: $\boldsymbol{J} = \begin{bmatrix} \mu & -\omega \\ \omega & \mu \end{bmatrix}$

eigenvalues: $\lambda_{1,2} = \mu \pm i\omega$



creation of periodic solutions

eigenvalues go as a complex conjugated pair through the imaginary axis

Transformation: $x_1 = r\cos\theta \qquad x_2 = r\sin\theta$

$\dot{r} = \mu r + \alpha r^3$ normal form pitchfork bifurcation

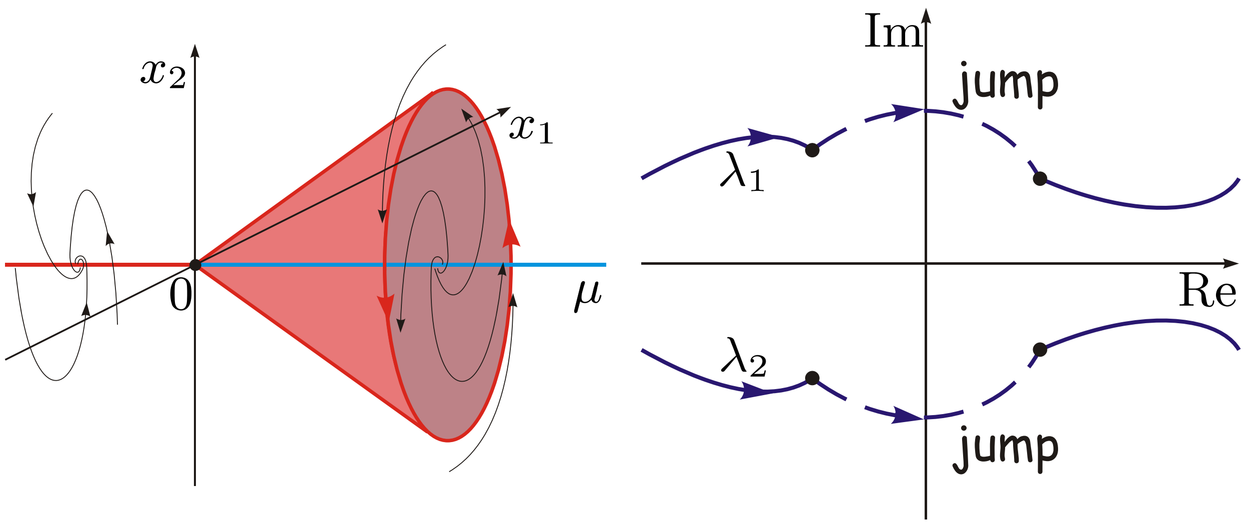
$\dot{\theta} = \omega + \beta r^2$

Non-smooth continuous counter part:

$$\dot{x}_1 = -x_1 - \omega x_2 + \frac{x_1}{\sqrt{x_1^2+x_2^2}}\left(\left|\sqrt{x_1^2+x_2^2} + \tfrac{1}{2}\mu\right| - \left|\sqrt{x_1^2+x_2^2} - \tfrac{1}{2}\mu\right|\right)$$

$$\dot{x}_2 = \omega x_1 - x_2 + \frac{x_2}{\sqrt{x_1^2+x_2^2}}\left(\left|\sqrt{x_1^2+x_2^2} + \tfrac{1}{2}\mu\right| - \left|\sqrt{x_1^2+x_2^2} - \tfrac{1}{2}\mu\right|\right)$$

somewhat strange



discontinuous Hopf bifurcation

after transformation in polar coordinates:

$$\dot{r} = -r + |r + \frac{1}{2}\mu| - |r - \frac{1}{2}\mu|$$ discontinuous pitchfork bif.

$$\dot{\theta} = \omega$$

An easier example of a discontinuous Hopf bifurcation

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_2 - \tfrac{3}{2}|x_2 - \mu| - x_1$$

equilibrium: $x_1 = -\dfrac{3}{2}|\mu|, \quad x_2 = 0$

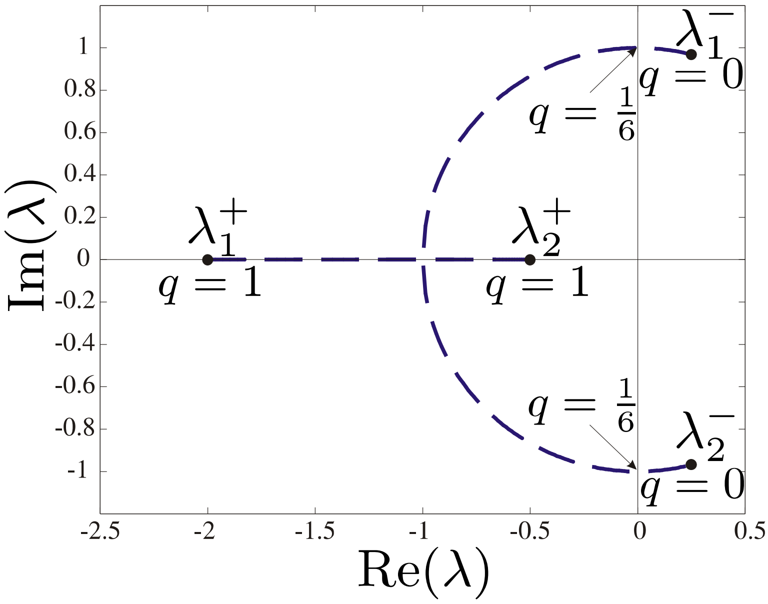
generalised Jacobian: $\boldsymbol{J}(\boldsymbol{x}, \mu) = \begin{bmatrix} 0 & 1 \\ -1 & -1 - \frac{3}{2}\,\mathrm{Sign}(x_2 - \mu) \end{bmatrix}$

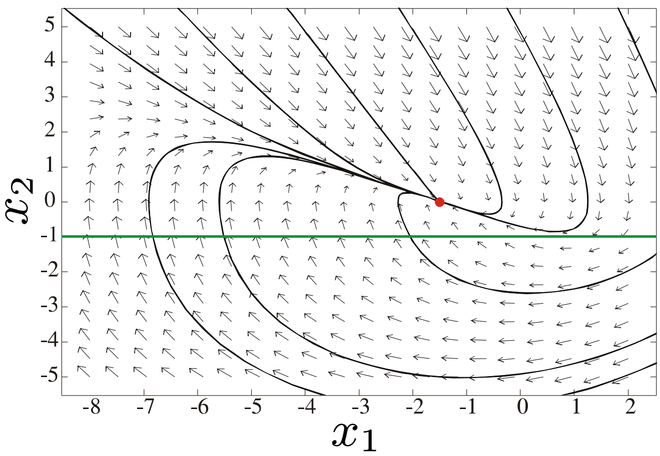
$\boldsymbol{J}(\boldsymbol{0}) = \{\boldsymbol{J}_q, q \in [0,1]\} \qquad \boldsymbol{J}_q = q\boldsymbol{J}_+ + (1-q)\boldsymbol{J}_-$

convex combination

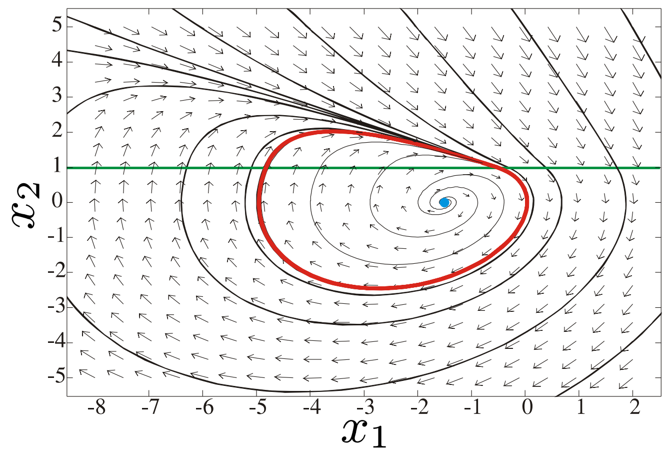
$$\boldsymbol{J}_+ = \begin{bmatrix} 0 & 1 \\ -1 & -\frac{5}{2} \end{bmatrix}$$

$$\boldsymbol{J}_- = \begin{bmatrix} 0 & 1 \\ -1 & \frac{1}{2} \end{bmatrix}$$





$\mu = -1$

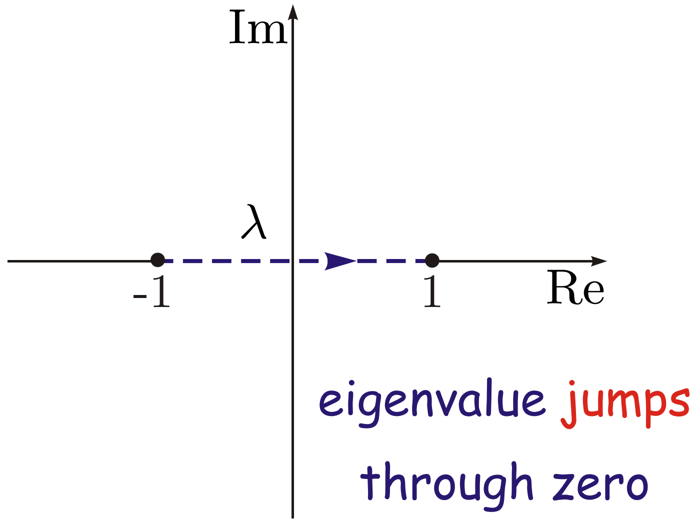


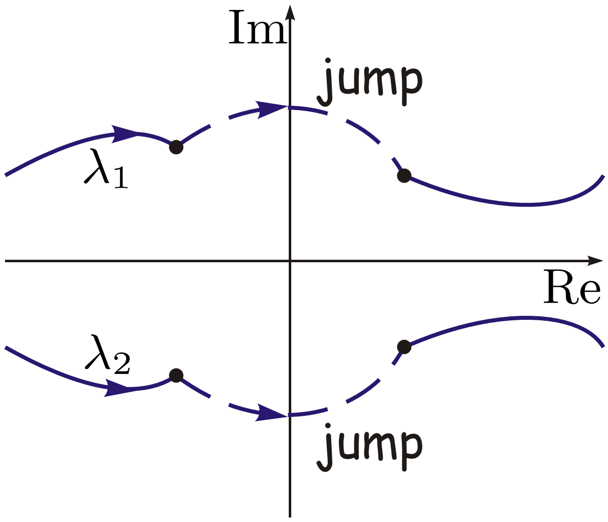
$\mu = 1$

5.5 Multiple Crossing Bifurcations

Single crossing bifurcation:

one (pair) of eigenvalue(s) of the set-valued generalised Jacobian crosses the imaginary axis once

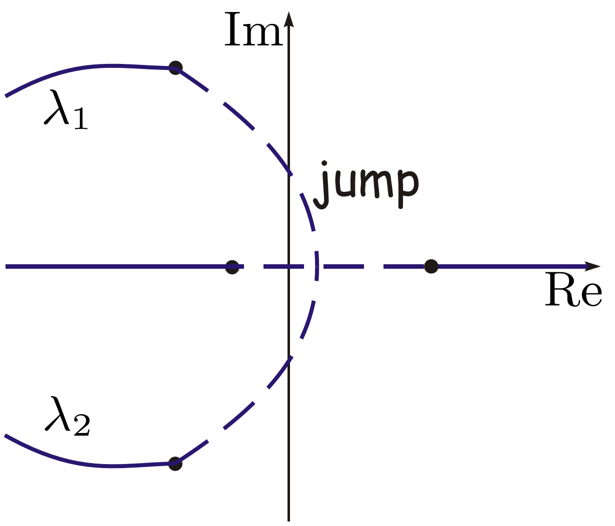


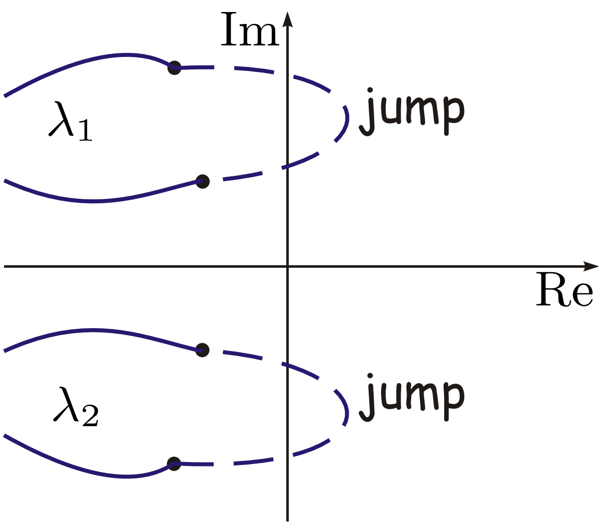


Much like classical bifurcations in smooth dynamical systems

Multiple crossing bifurcation:

one (pair) of eigenvalue(s) of the set-valued generalised Jacobian crosses the imaginary axis more than once





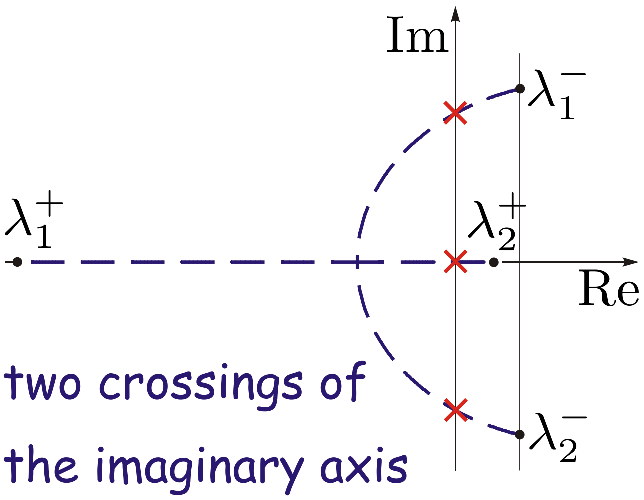
Much more complicated than single crossing bifurcations

Combined Hopf and turning point behaviour

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + |x_1 + \mu| - |x_1 - \mu| - x_2 - |x_2 + \mu| + |x_2 - \mu|$$

generalised Jacobian:

$$\boldsymbol{J}(\boldsymbol{x}, \mu) = \begin{bmatrix} 0 & 1 \\ J_{21} & J_{22} \end{bmatrix} \quad \begin{aligned} J_{21} &= -1 + \mathrm{Sign}(x_1 + \mu) - \mathrm{Sign}(x_1 - \mu) \\ J_{22} &= -1 - \mathrm{Sign}(x_2 + \mu) + \mathrm{Sign}(x_2 - \mu) \end{aligned}$$

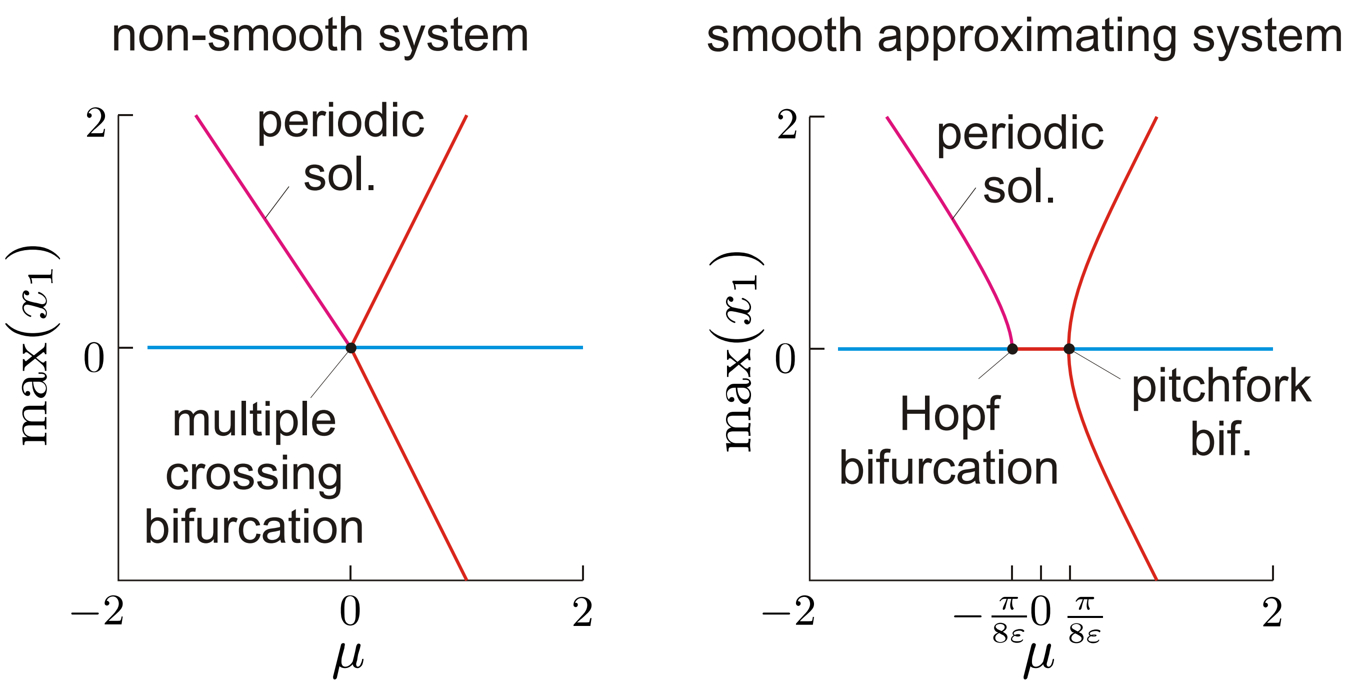


$$\boldsymbol{J}_-^{\mathrm{tr}} = \boldsymbol{J}(\boldsymbol{0}, \mu < 0) = \begin{bmatrix} 0 & 1 \\ -3 & 1 \end{bmatrix}$$

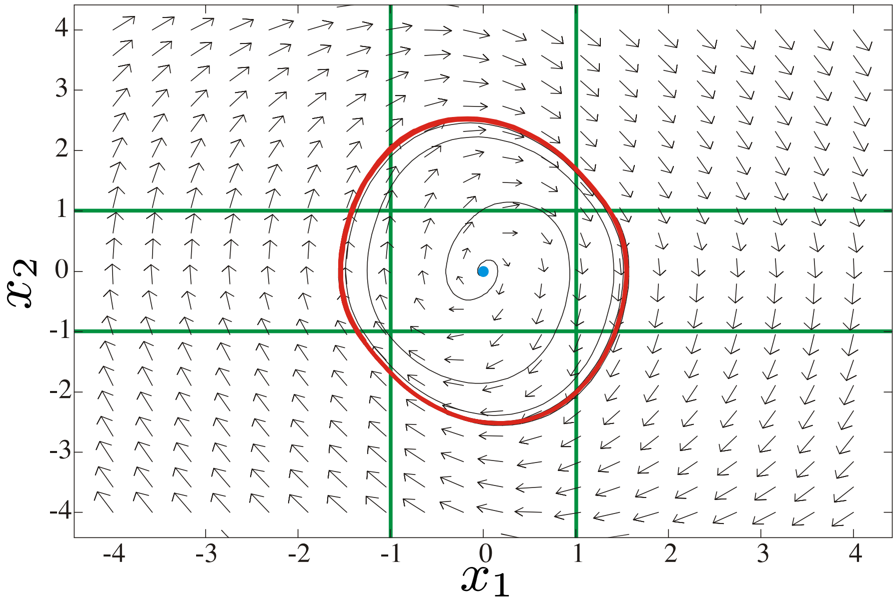
$$\lambda_{1,2}^- = \frac{1}{2} \pm i\frac{1}{2}\sqrt{11}$$

$$\boldsymbol{J}_+^{\mathrm{tr}} = \boldsymbol{J}(\boldsymbol{0}, \mu > 0) = \begin{bmatrix} 0 & 1 \\ 1 & -3 \end{bmatrix}$$

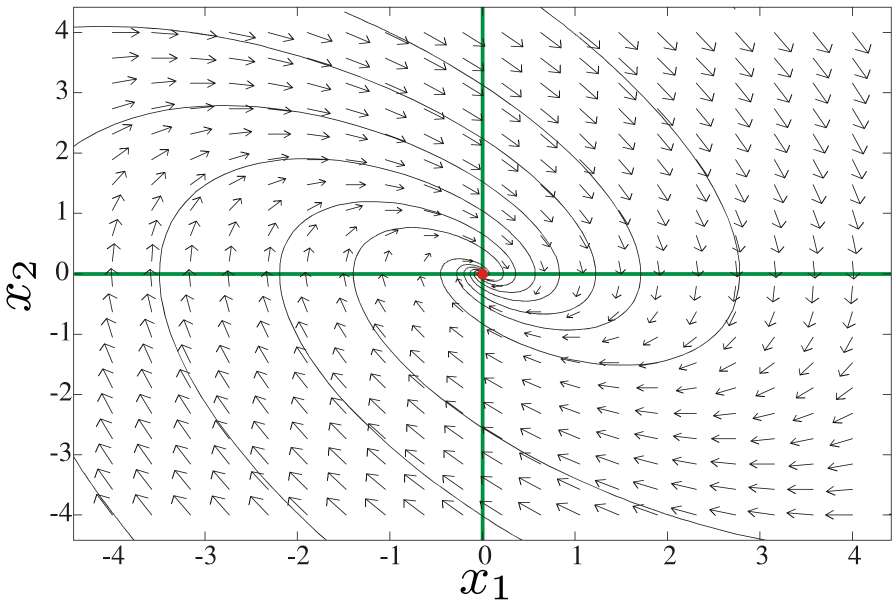
$$\lambda_{1,2}^+ \approx \{0.30, -3.30\}$$



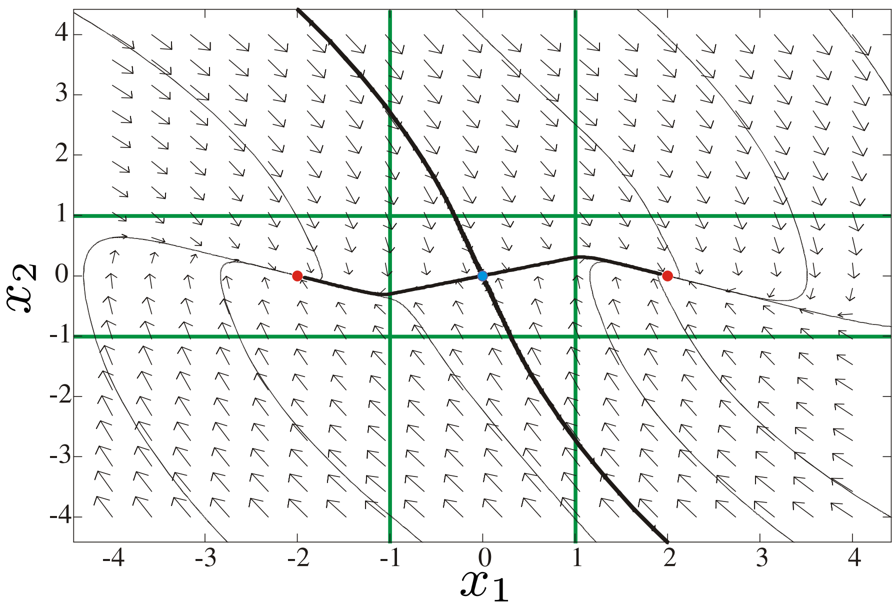
$\mu = -1$



$\mu = 0$



$\mu = 1$



6 Bifurcations of Fixed Points

Nonlinear map: $\boldsymbol{P} : \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\boldsymbol{y}_{i+1} = \boldsymbol{P}(\boldsymbol{y}_i)$$

Fixed point: $\boldsymbol{y}^* = \boldsymbol{P}(\boldsymbol{y}^*)$

6.1 Linearisation around a Fixed Point

Perturbation:

$$\boldsymbol{y}_i = \boldsymbol{y}^* + \boldsymbol{x}_i \qquad \|\boldsymbol{x}_i\| \ll 1$$

Substitution and linearisation:

$$\begin{aligned}\boldsymbol{y}^* + \boldsymbol{x}_{i+1} &= \boldsymbol{P}(\boldsymbol{y}^* + \boldsymbol{x}_i) \\ &= \boldsymbol{P}(\boldsymbol{y}^*) + \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right|_{\boldsymbol{y}^*} \boldsymbol{x}_i + H.O.T.\end{aligned}$$

Study the linear map:

$$\Longrightarrow \boldsymbol{x}_{i+1} = \boldsymbol{A}\boldsymbol{x}_i \qquad \boldsymbol{A} = \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right|_{\boldsymbol{y}^*}$$

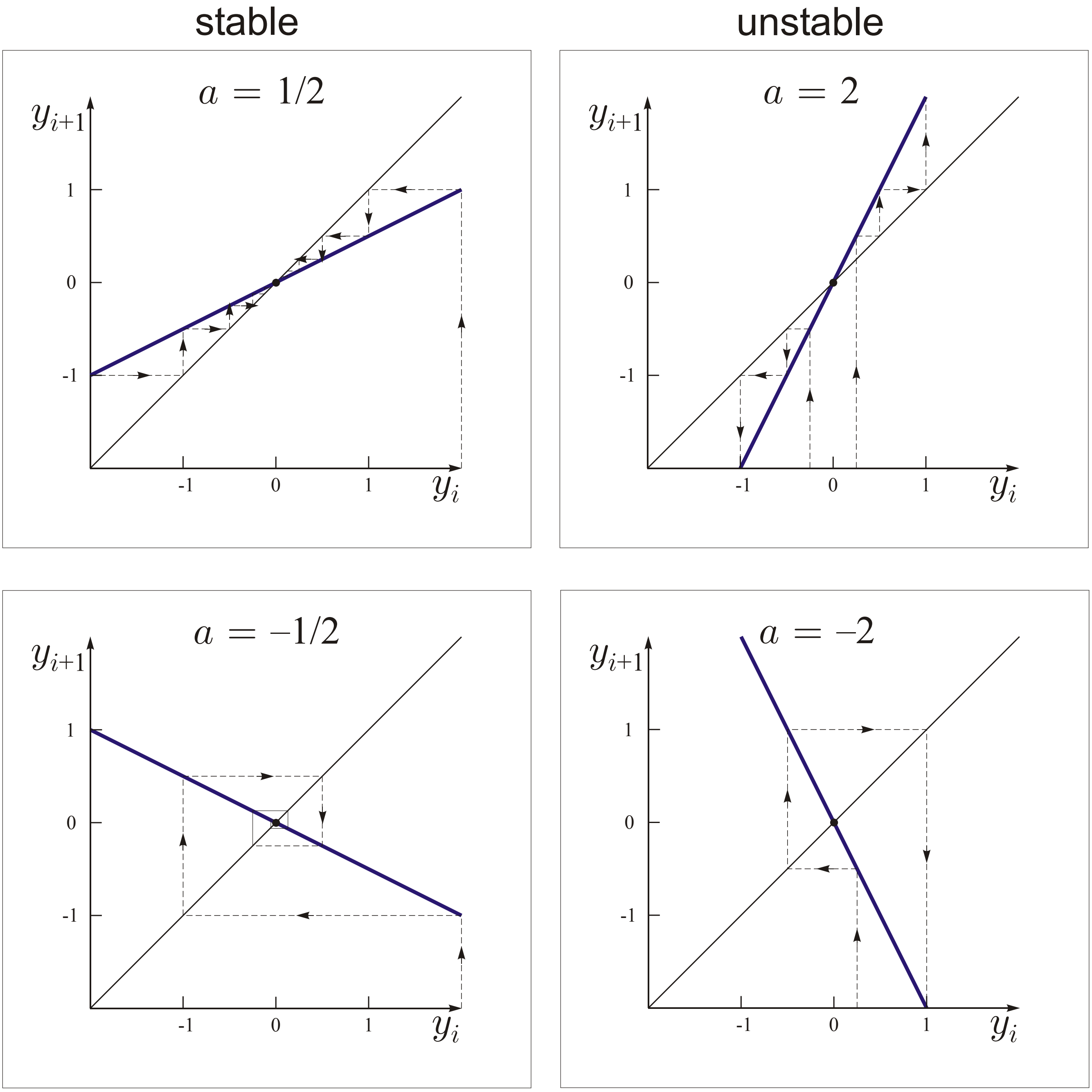
6.2 One-dimensional Linear Maps

$$y_{i+1} = a y_i \qquad a \in \mathbb{R} \quad y^* = 0$$

$$y_1 = a y_0$$

$$y_2 = a y_1 = a^2 y_0$$

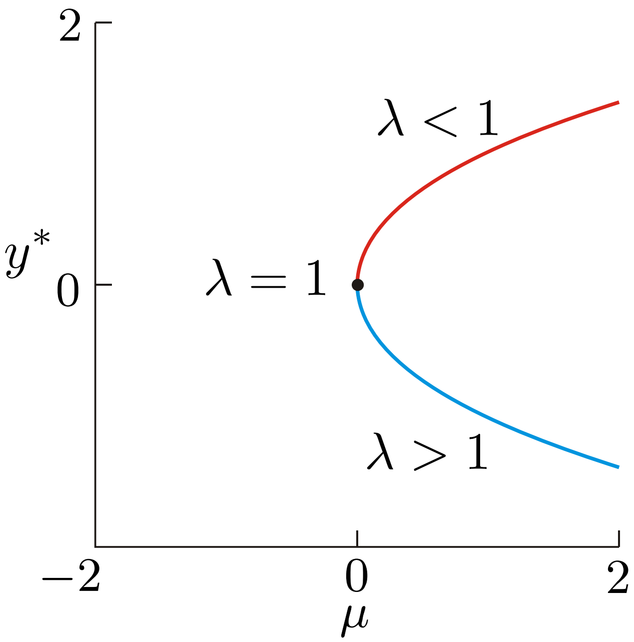
$$y_n = a^n y_0$$

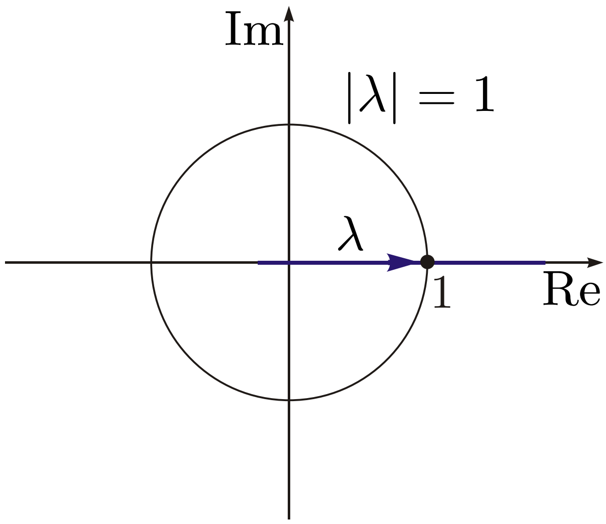


6.3 Turning Point Bifurcation

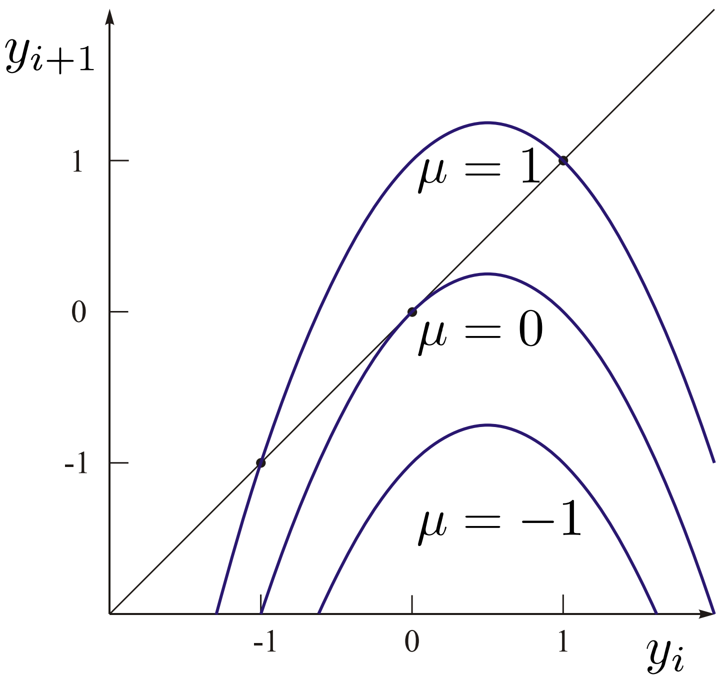
$$y_{i+1} = y_i + \mu - y_i^2$$

fixed points: $y^* = y^* + \mu - (y^*)^2 \implies y^*_{1,2} = \pm\sqrt{\mu}$





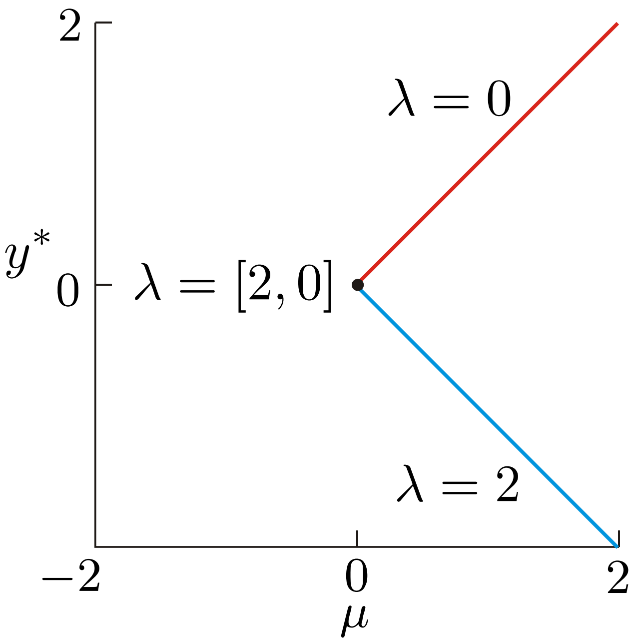
eigenvalue goes through unit circle



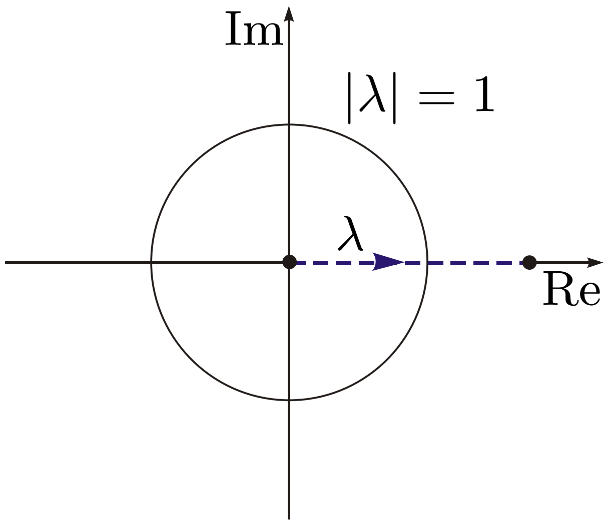
non-smooth continuous map:

$$y_{i+1} = y_i + \mu - |y_i|$$

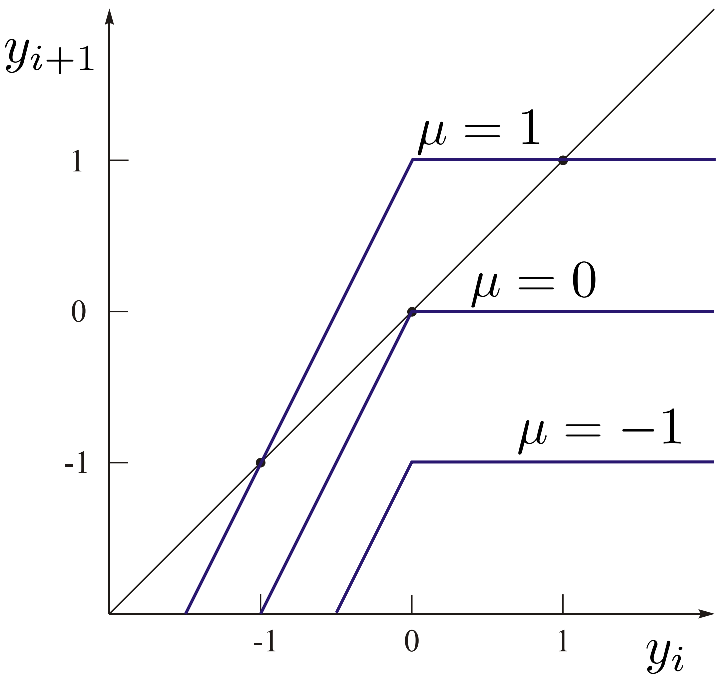
fixed points: $y^* = y^* + \mu - |y^*| \implies y^*_{1,2} = \pm\mu \geq 0$



discontinuous turning-point bif.



eigenvalue jumps through unit circle



similarly:

discontinuous pitchfork

&

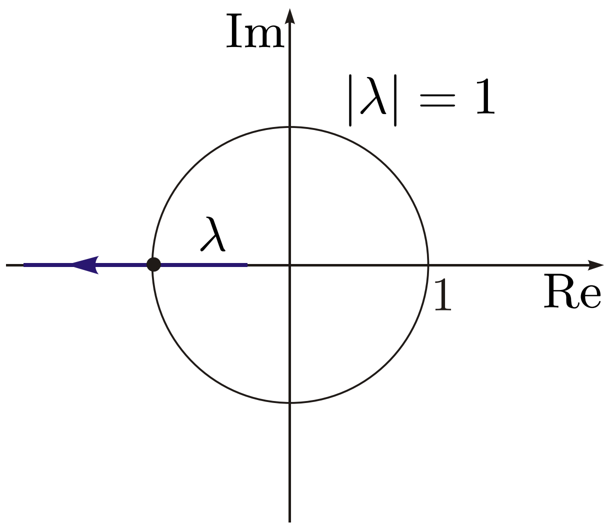
discontinuous transcritical bifurcation

6.4 Flip Bifurcation

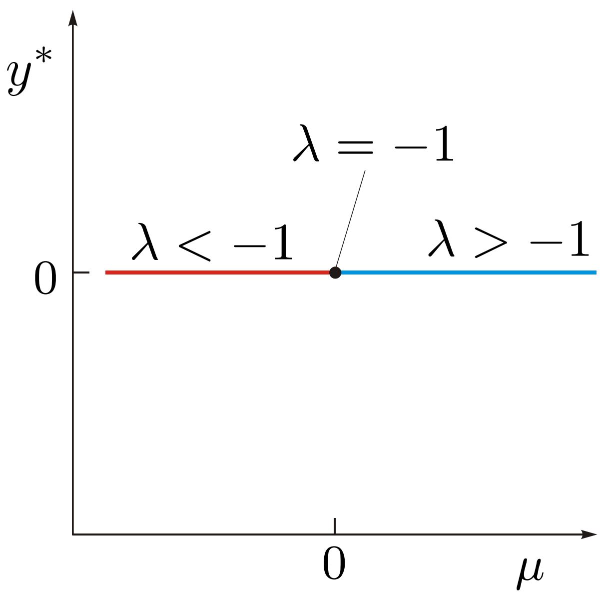
$$y_{i+1} = -(1+\mu)y_i + y_i^3 \qquad |y_i| < 1 \quad |\mu| < 1$$

$y^* = 0$ unique fixed point for $|\mu| < 1$

eigenvalue: $\lambda = -(1+\mu)$

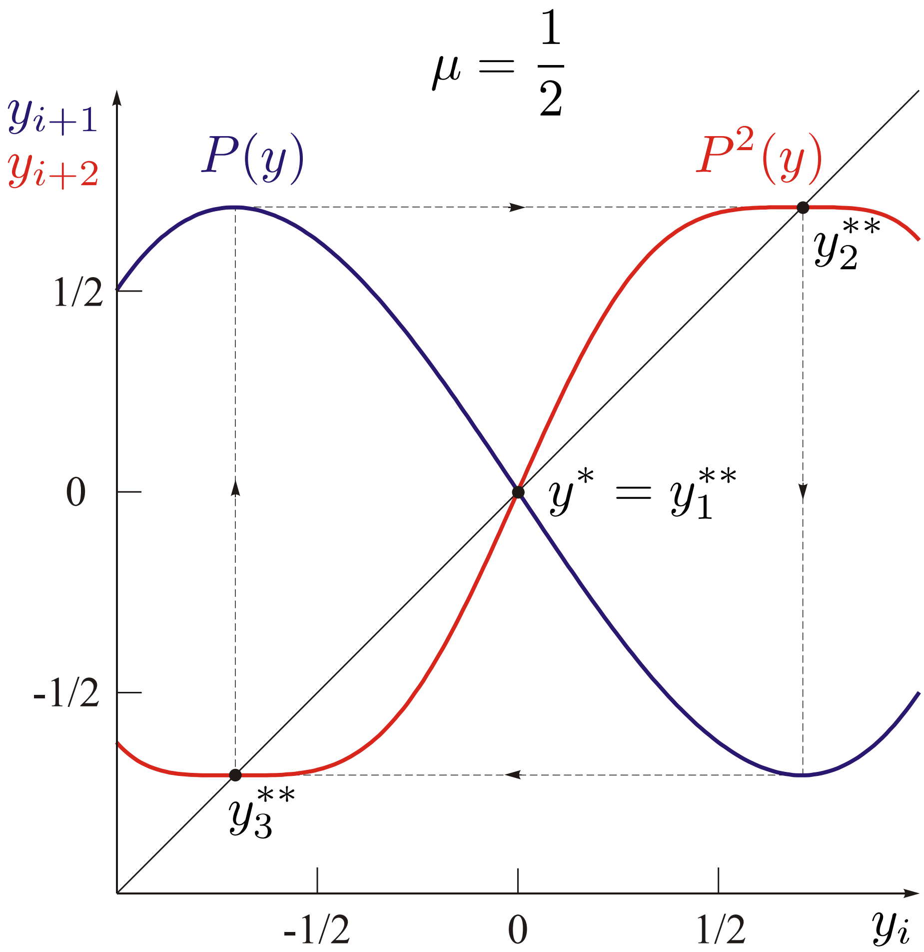


eigenvalue goes through -1

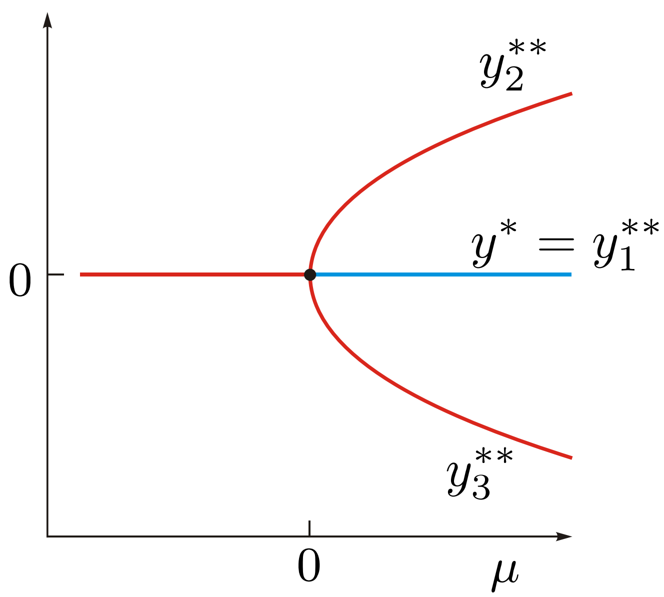


stability is exchanged...

is there a bifurcation?

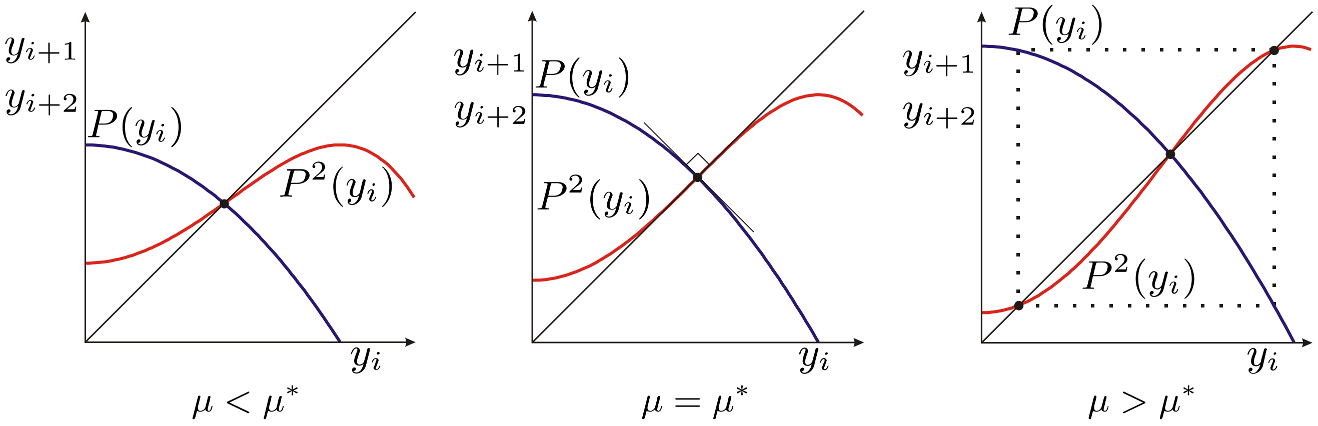


flip!

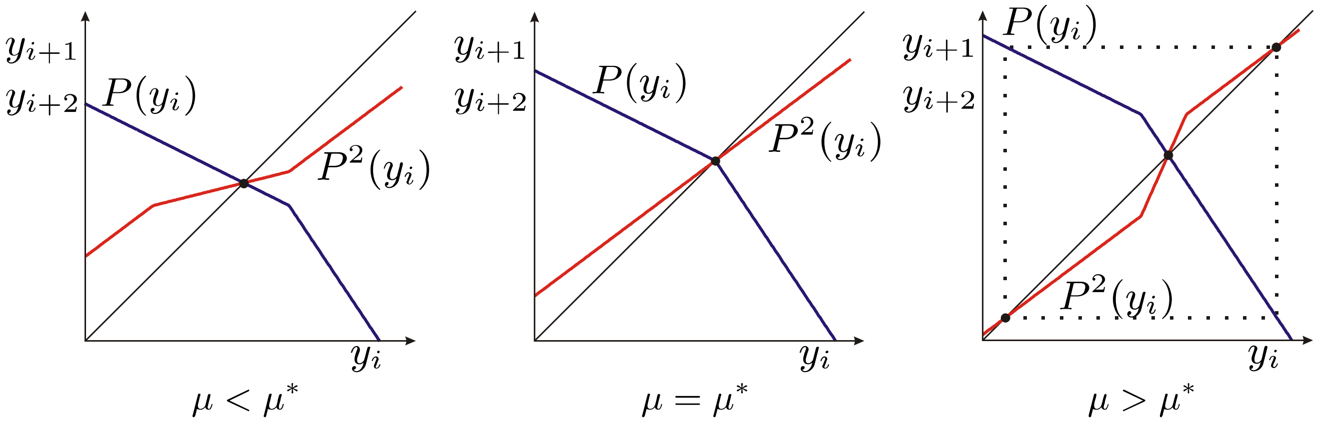


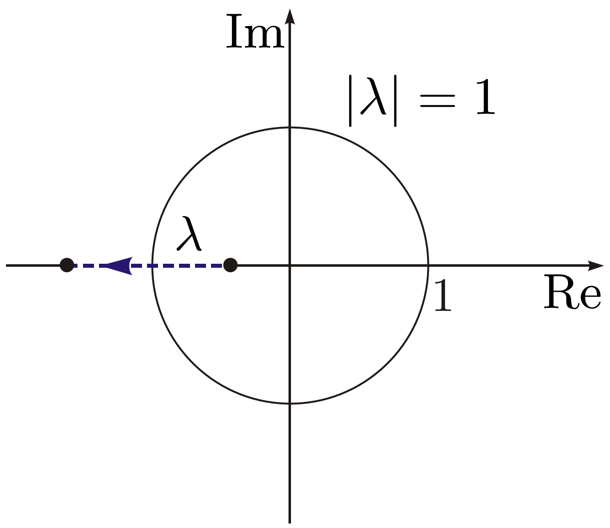
A flip bifurcation of the period-1 map is a pitchfork bifurcation of the period-2 map

continuous flip bifurcation



discontinuous flip bifurcation



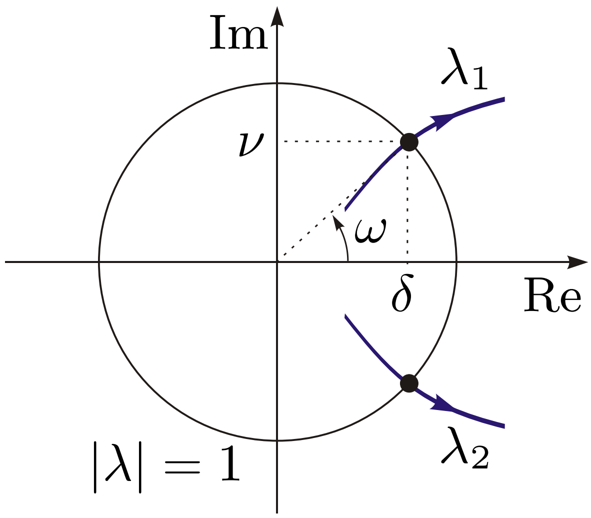


eigenvalue jumps

through -1

6.5 Naimark-Sacker Bifurcation (2nd Hopf bifurcation)

$$x_{i+1} = \delta x_i - \nu y_i + \gamma x_i (x_i^2 + y_i^2)$$
$$y_{i+1} = \nu x_i + \delta y_i + \gamma y_i (x_i^2 + y_i^2)$$



$$\omega = \arctan\left(\frac{\nu}{\delta}\right)$$

complex pair of eigenvalues crosses the unit cricle

If ω and 2π are commensurate, $k\omega = n \cdot 2\pi, \quad k, n \in \mathbb{N}$
then a period-k solution is created/destroyed.

If ω and 2π are incommensurate,
then a quasi-periodic solution is created/destroyed.

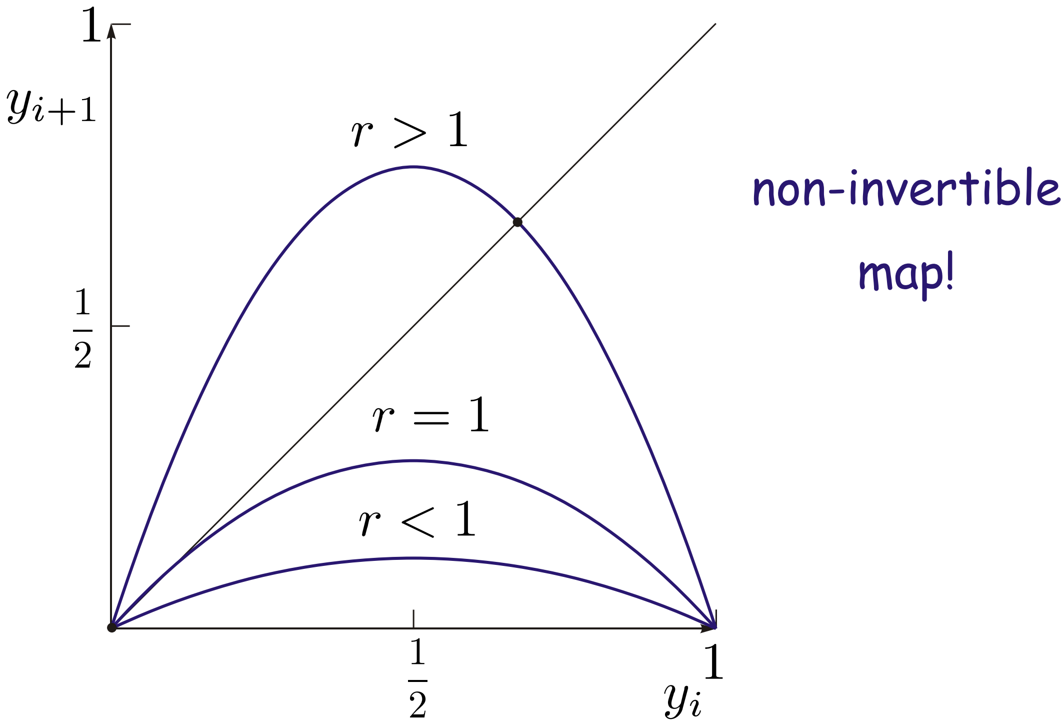
6.5 The Logistic Map

A simple nonlinear map can have very complicated dynamics

$$y_{i+1} = P(y_i) = r y_i (1 - y_i) \qquad 0 \le y \le 1 \quad r \ge 1$$

Fixed points: $y^* = r y^* (1 - y^*)$

$$y_1^* = 0 \ \ \forall r \quad \text{and} \quad y_2^* = 1 - \frac{1}{r} \in [0,1] \quad \text{for } r \ge 1$$

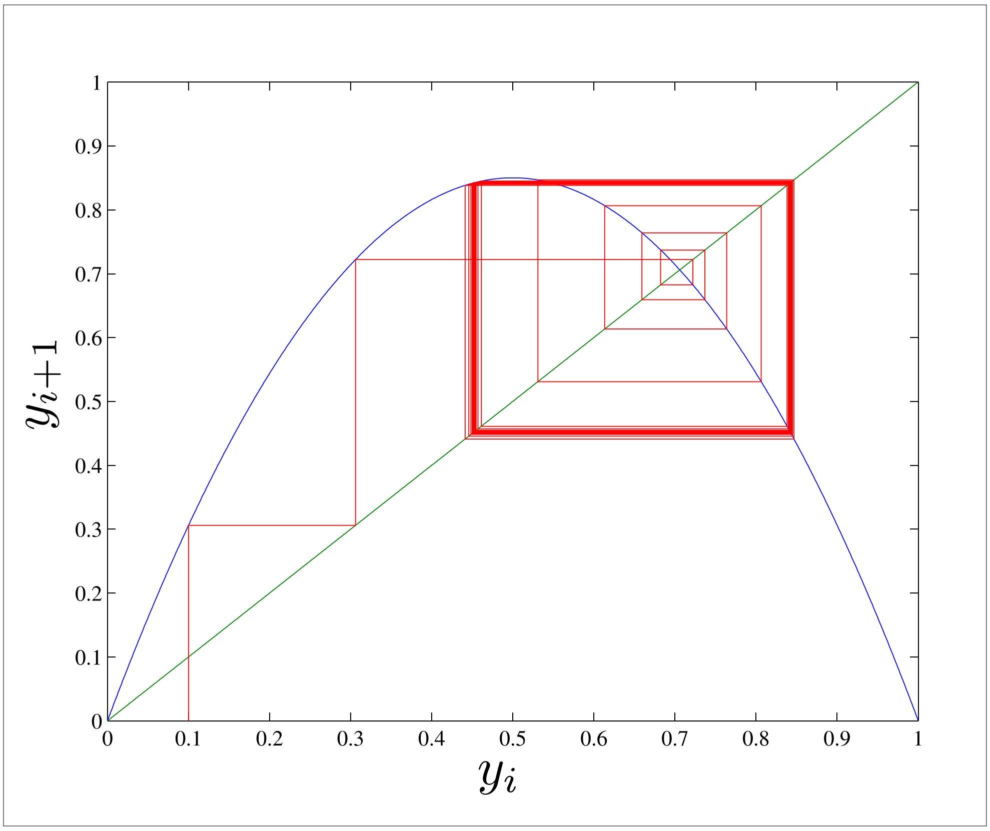


stability: $\dfrac{\partial P}{\partial y} = r - 2ry$

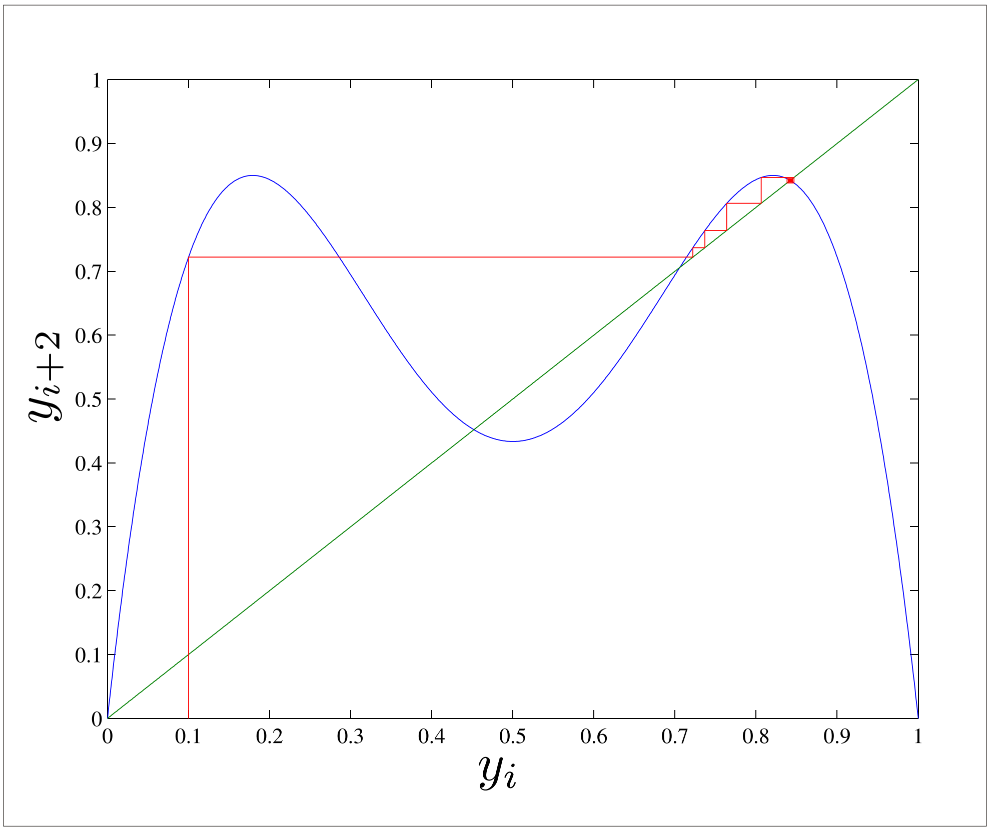
$$\lambda_1 = r \qquad\qquad \lambda_2 = -r + 2$$

y_2^* stable for $1 < r < 3$

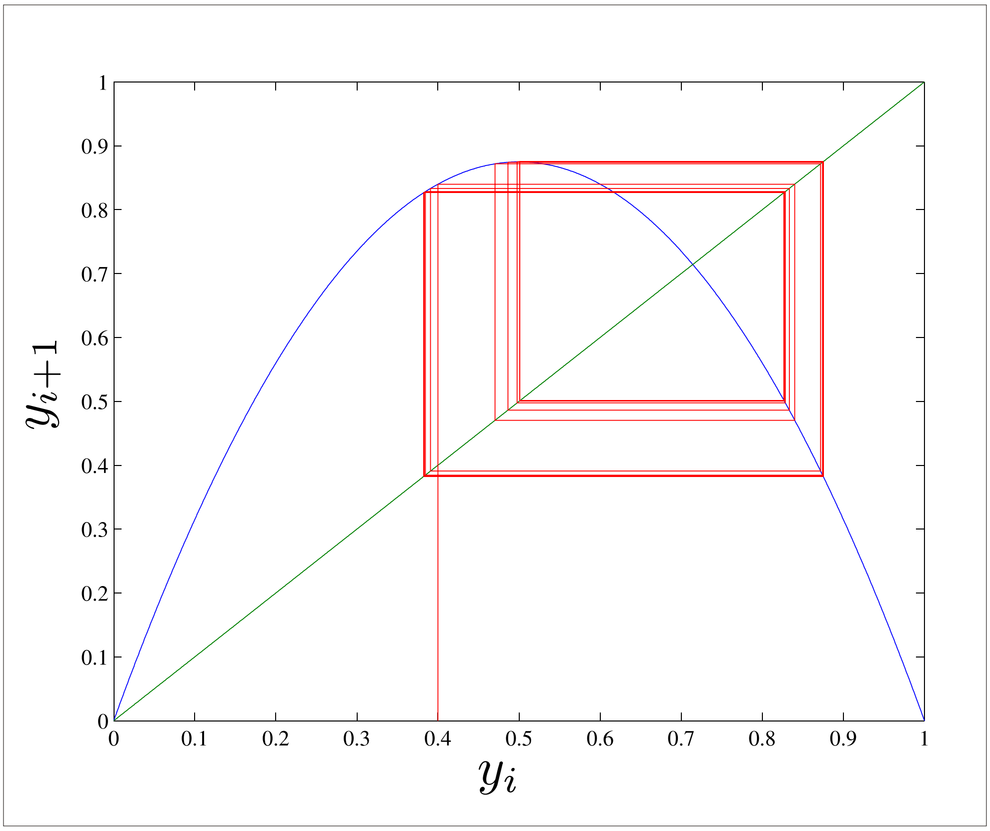
λ_2 goes through -1 at $r = 3 \Longrightarrow$ flip bifurcation



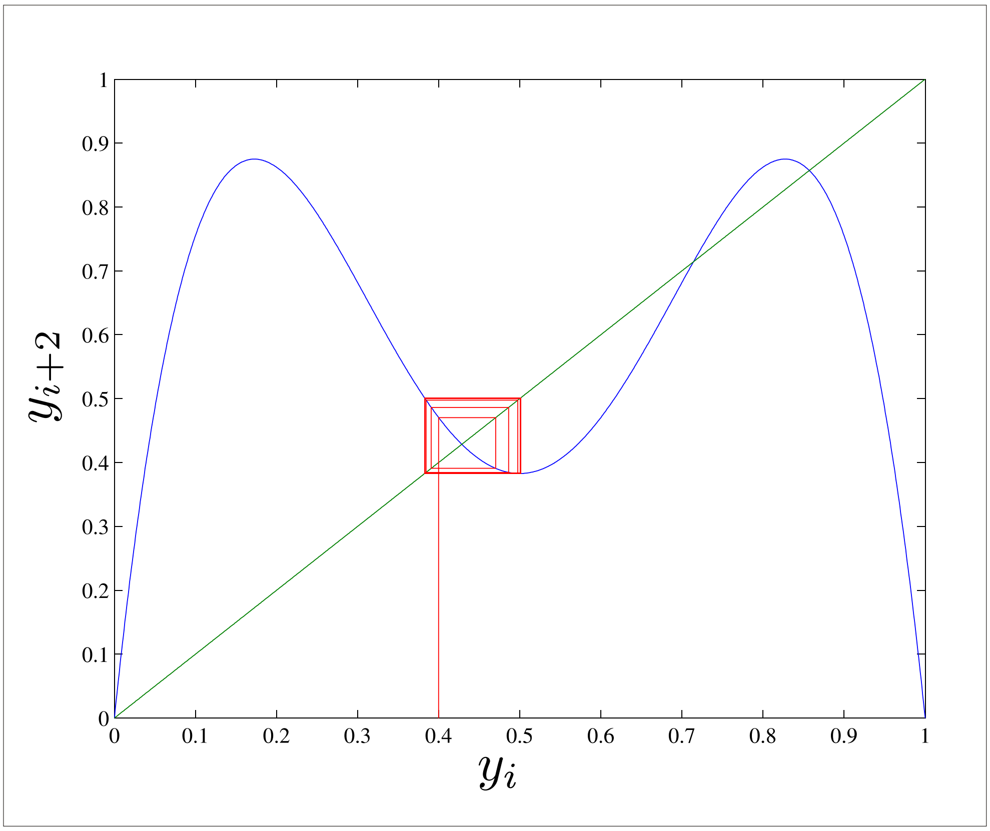
Logistic map, $r = 3.4$



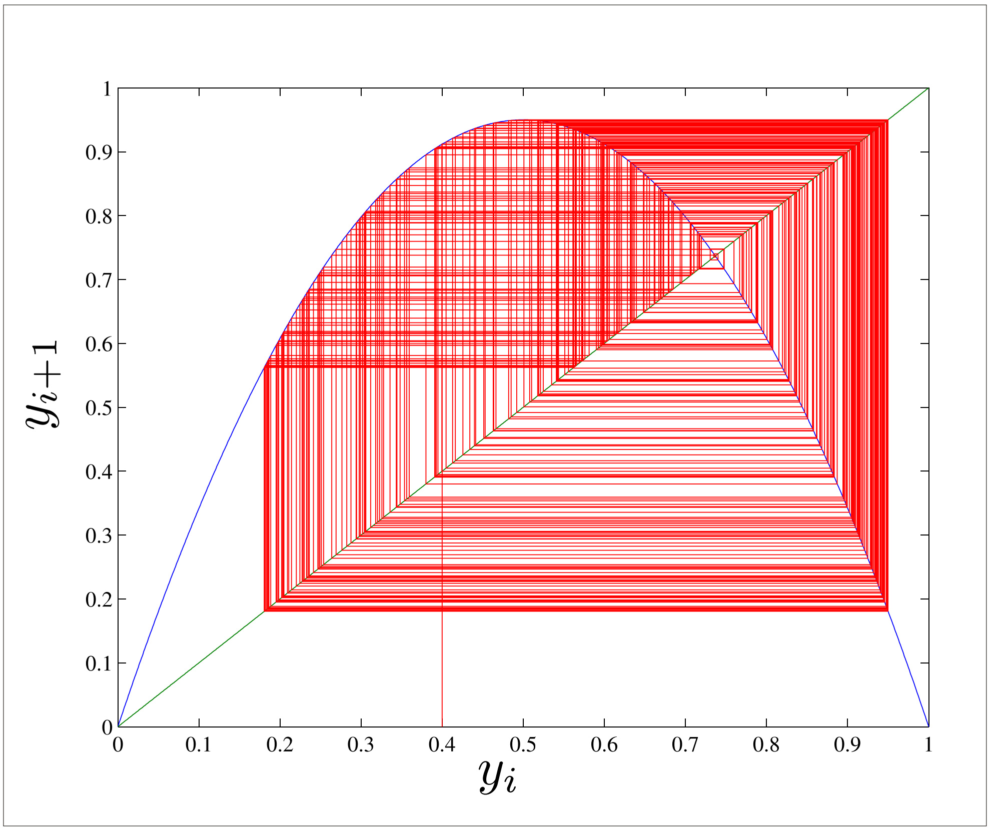
2nd iterated Logistic map, $r = 3.4$



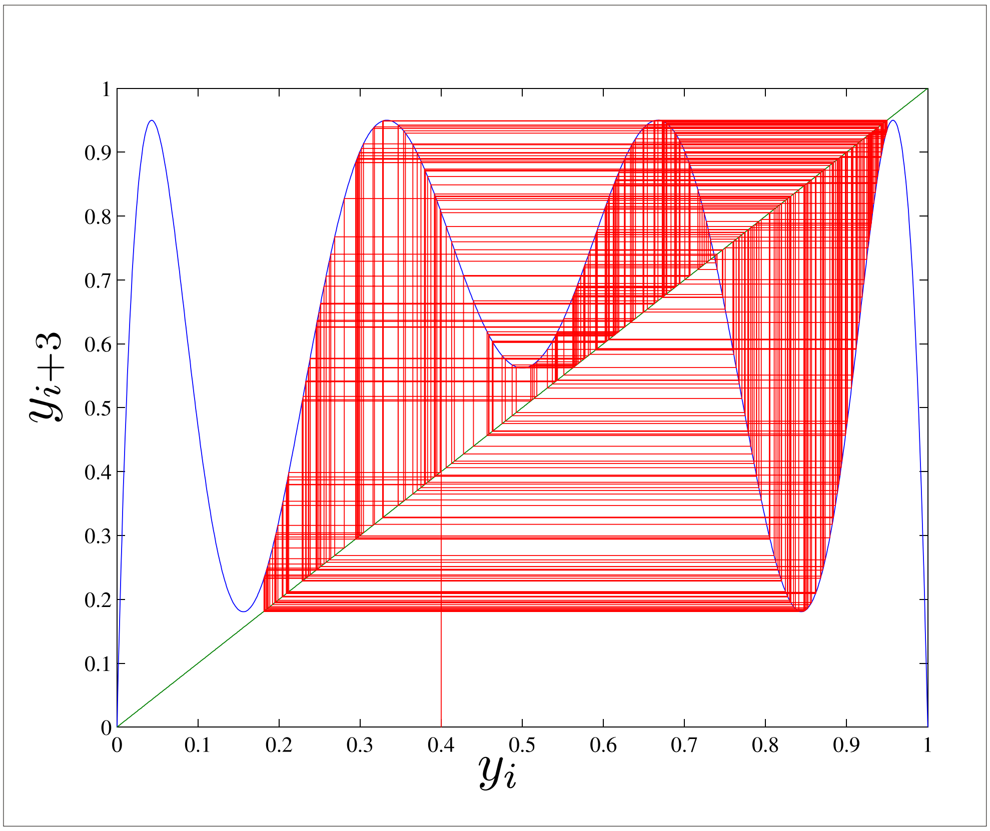
Logistic map, $r = 3.5$



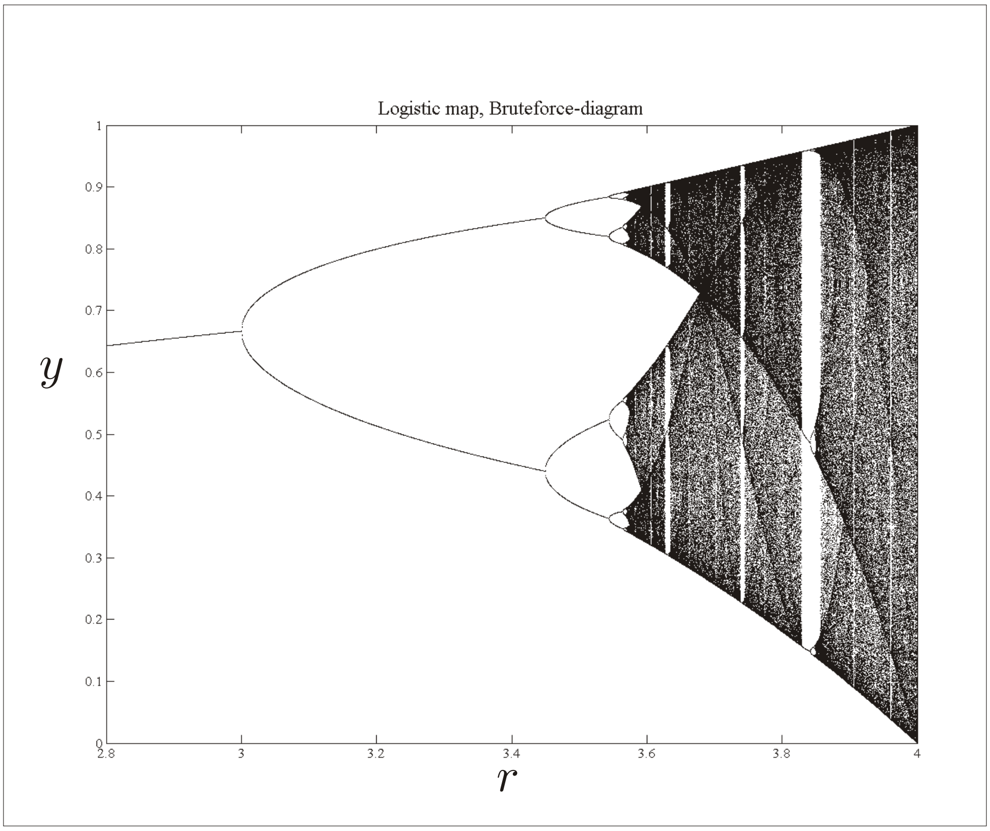
2nd iterated Logistic map, $r = 3.5$



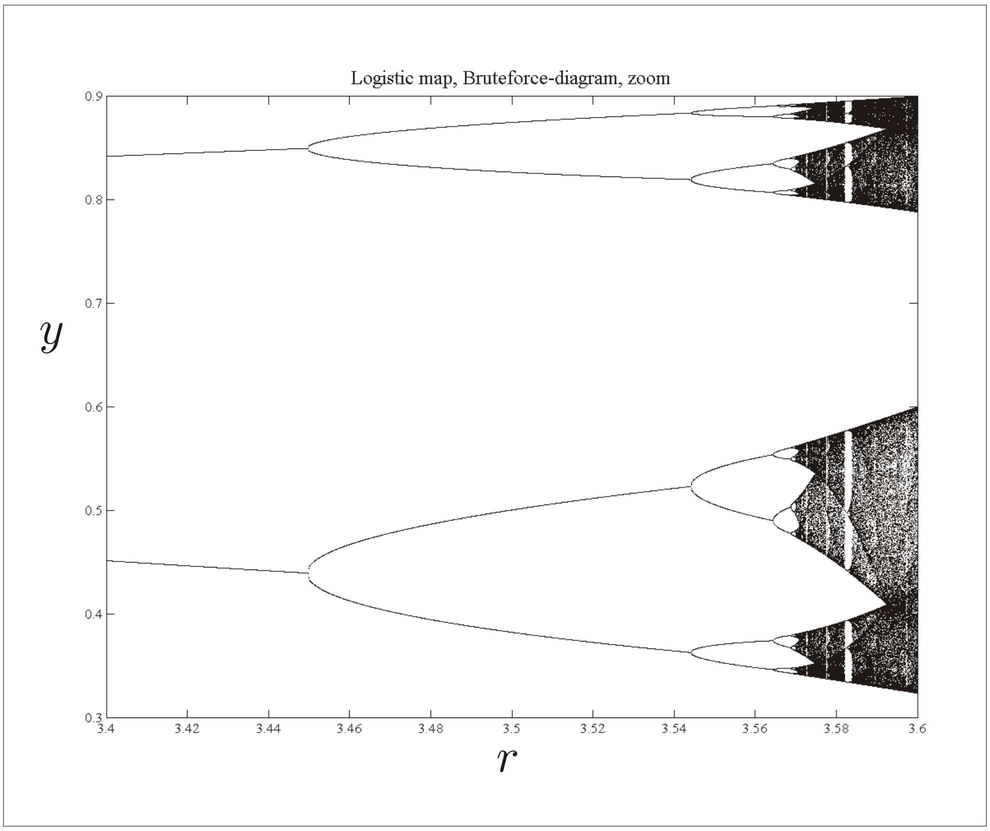
Logistic map, $r = 3.8$



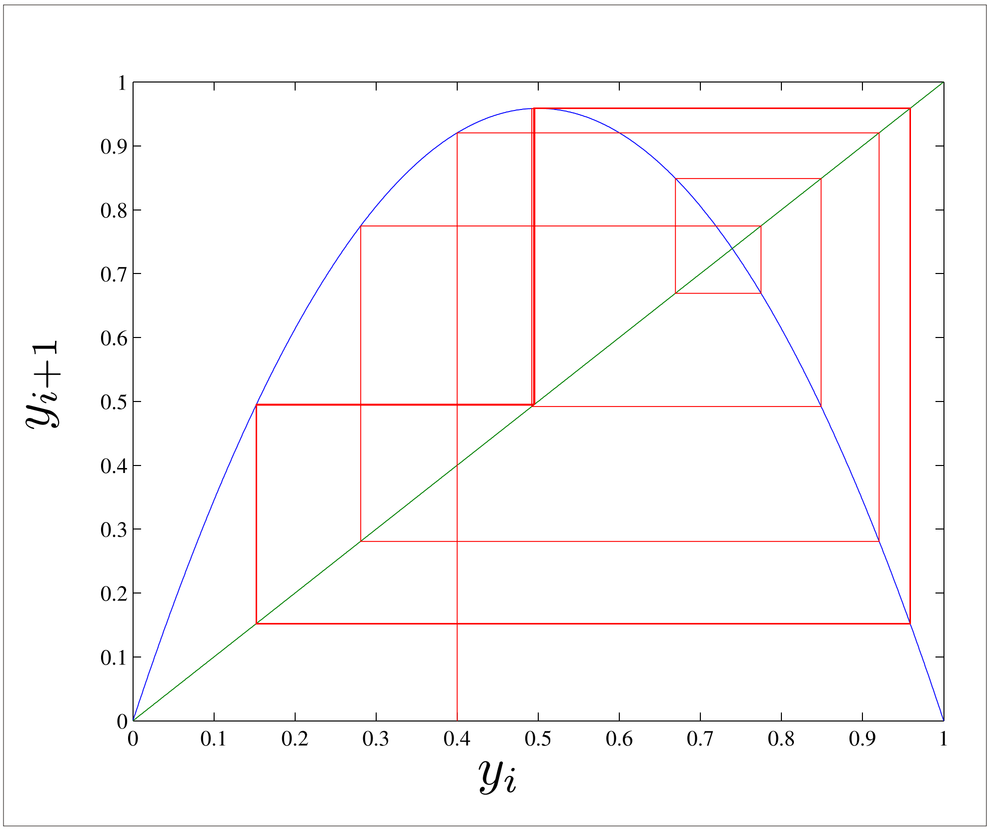
3rd iterated Logistic map, $r = 3.8$



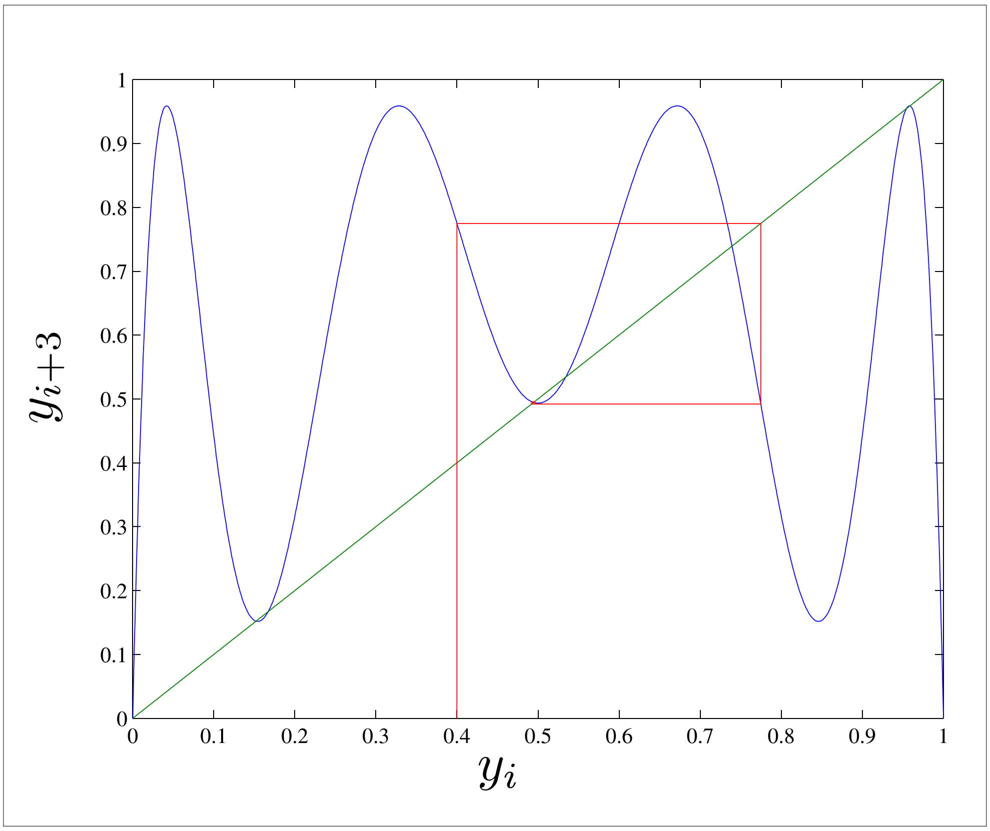
Logistic map, brute force diagram



Logistic map, brute-force diagram zoom

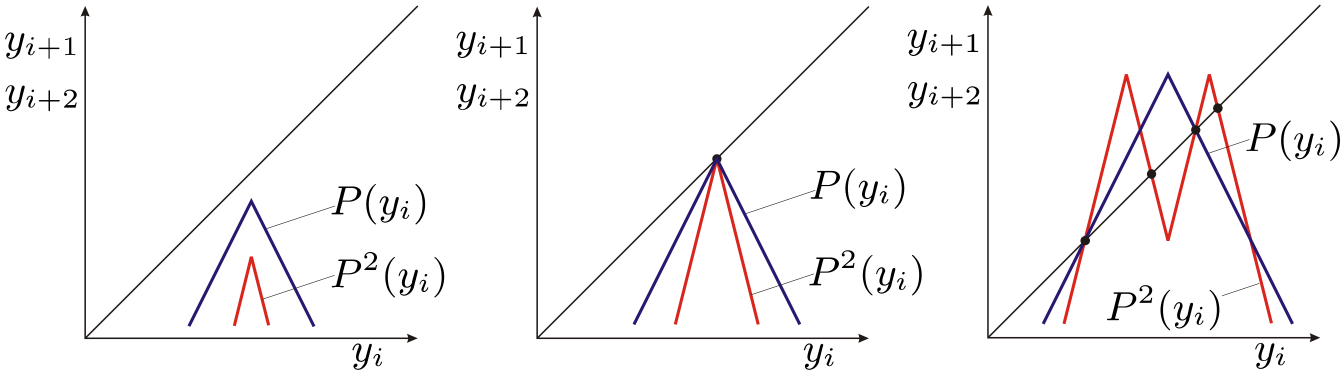


Logistic map, $r = 3.835$

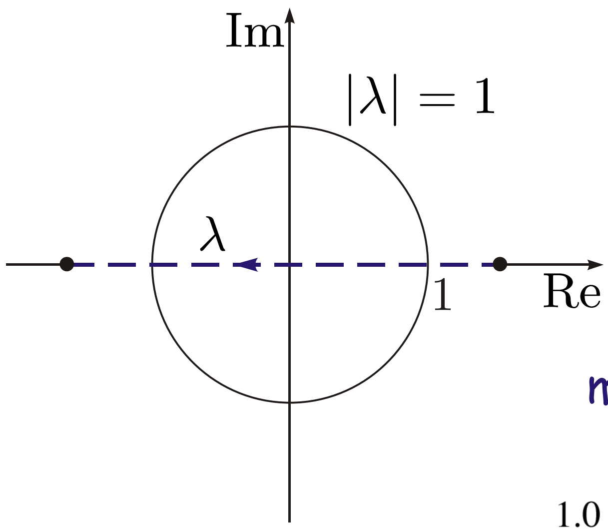


3rd iterated Logistic map, $r = 3.835$

6.6 The Tent Map



$\mu < \mu^*$ $\qquad$ $\mu = \mu^*$ $\qquad$ $\mu > \mu^*$



eigenvalue jumps

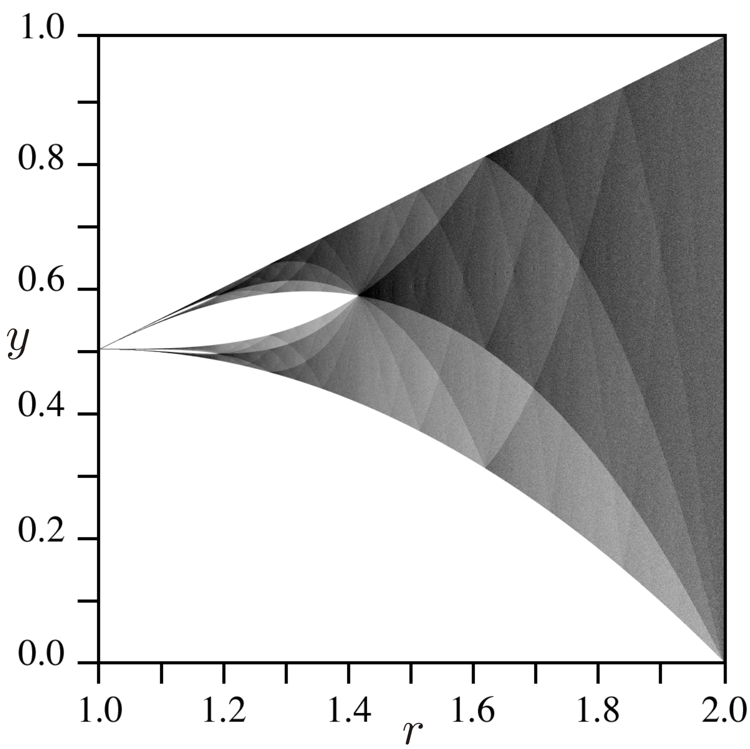
through +1 and -1

multiple crossing bifurcation

creation of infinitely many unstable branches

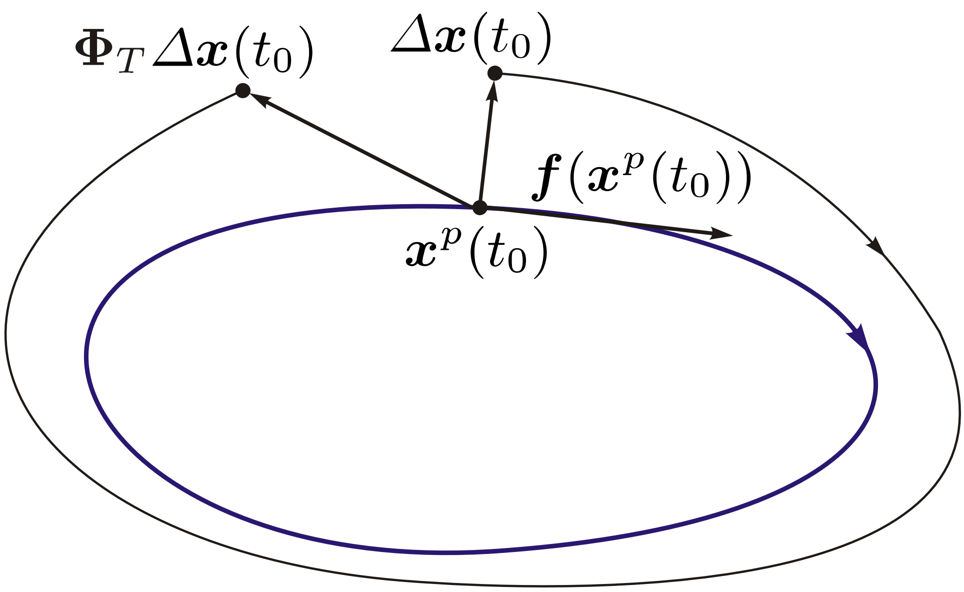
⇒instant chaos

brute-force bifurcation diagram



7 Bifurcations of Periodic Solutions

7.1 Stability of Periodic Solutions



Fundamental solution matrix $\boldsymbol{\Phi}(t)$ describes the influence of small perturbations:

$$\Delta \boldsymbol{x}(t) = \boldsymbol{\Phi}(t)\Delta \boldsymbol{x}(t_0) + H.O.T.$$

Smooth systems: $$\dot{\boldsymbol{\Phi}}(t) = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_p(t)} \boldsymbol{\Phi}(t) \qquad \boldsymbol{\Phi}(t_0) = \boldsymbol{I}$$

$\boldsymbol{\Phi}(t)$ is a local linearsation around the periodic solution!

Monodromy matrix: $\boldsymbol{\Phi}_T = \boldsymbol{\Phi}(T)$

Floquet multipliers: $(\lambda_1, \ldots, \lambda_n) = \operatorname{eig} \boldsymbol{\Phi}_T$

describe the exponential convergence/divergence of the perturbed solution w.r.t. the periodic solution

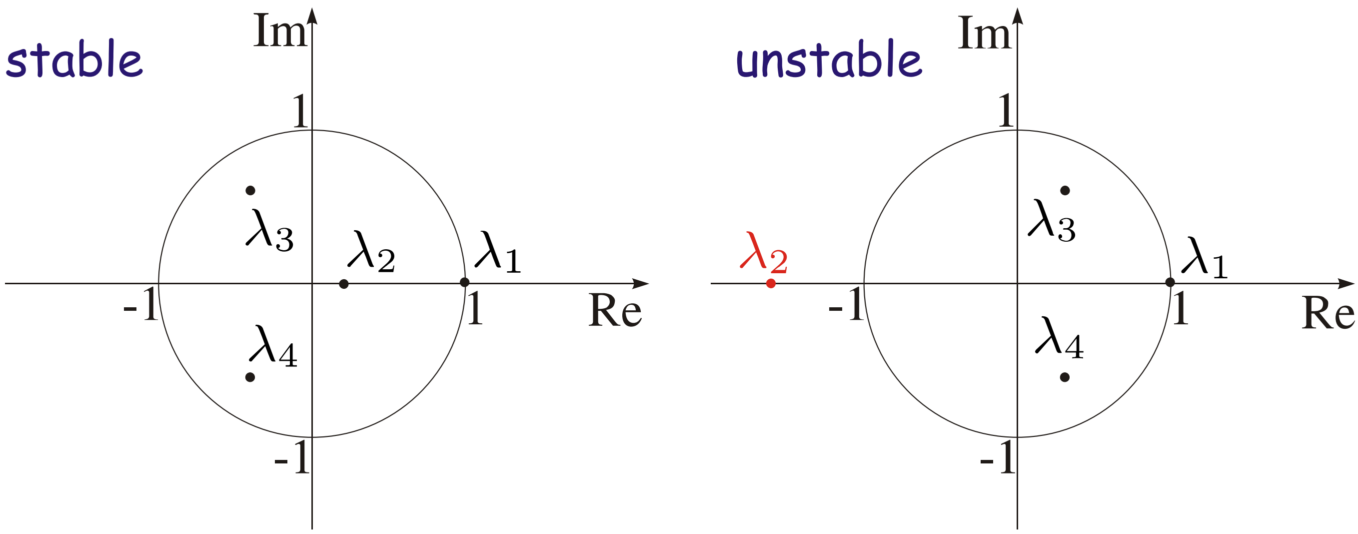
Autonomous systems: phase of the periodic solution is not fixed

if $\Delta \boldsymbol{x}(t_0) = \boldsymbol{f}(\boldsymbol{x}^p(t_0))$ then

$$\Delta \boldsymbol{x}(t_0) = \Delta \boldsymbol{x}(t_0 + T) = \boldsymbol{\Phi}_T \Delta \boldsymbol{x}(t_0)$$

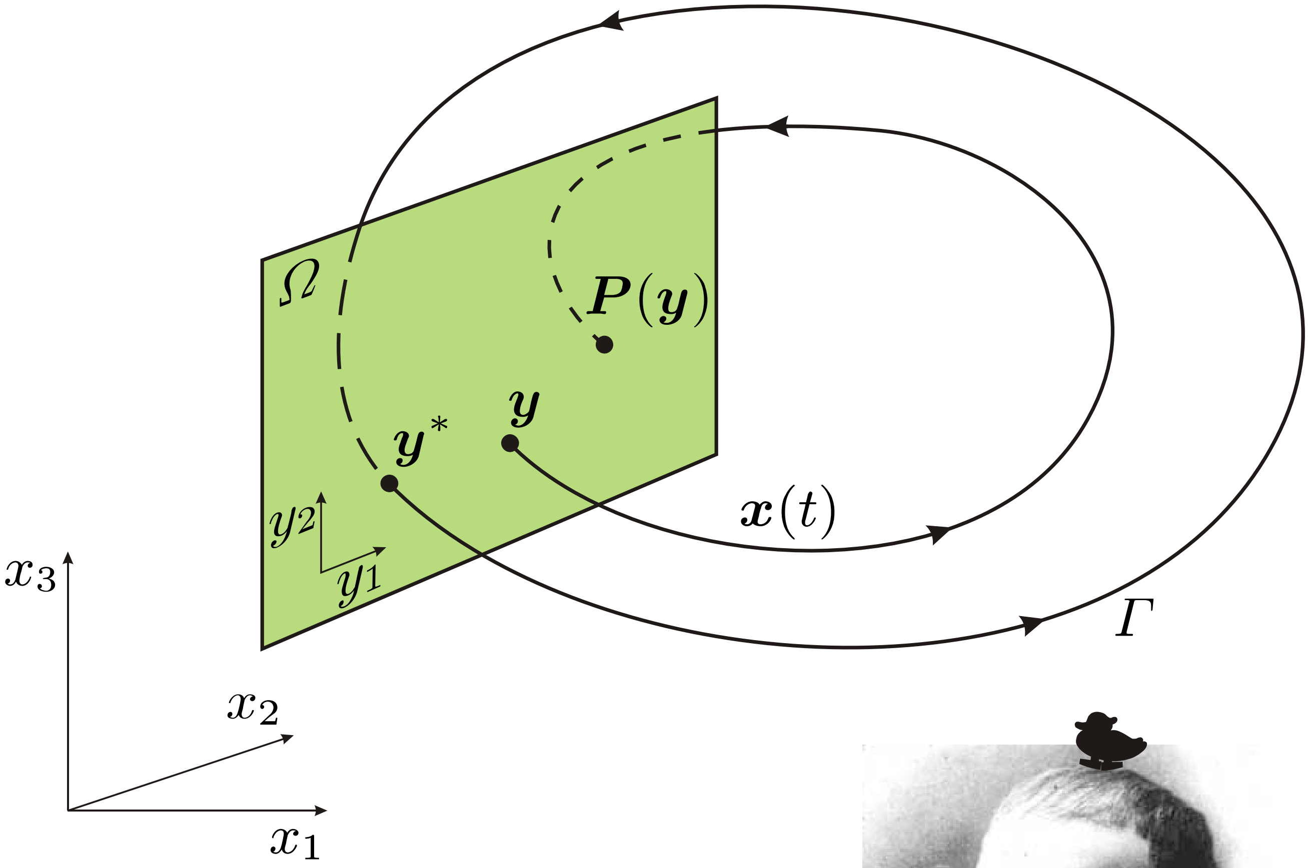
$$\boldsymbol{f}(\boldsymbol{x}^p(t_0)) = \boldsymbol{\Phi}_T \boldsymbol{f}(\boldsymbol{x}^p(t_0))$$

$\boldsymbol{f}(\boldsymbol{x}^p(t_0))$ is an eigenvector of $\boldsymbol{\Phi}_T$ with eigenvalue $\lambda_1 = 1$



If all Floquet multipliers, which are not associated with the freedom of phase, are within the unit circle, then the periodic solution is stable and attractive.

7.2 The Poincaré Map



Poincaré map $\boldsymbol{y}_{i+1} = \boldsymbol{P}(\boldsymbol{y}_i)$

periodic solution $\boldsymbol{y}^* = \boldsymbol{P}(\boldsymbol{y}^*)$

period-2 solution $\boldsymbol{y}^* = \boldsymbol{P}(\boldsymbol{P}(\boldsymbol{y}^*))$

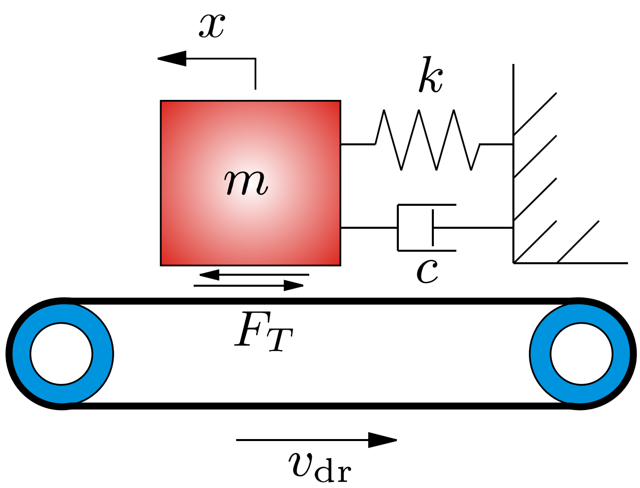
The Poincaré map transforms a continuous-time system into a discrete-time system.

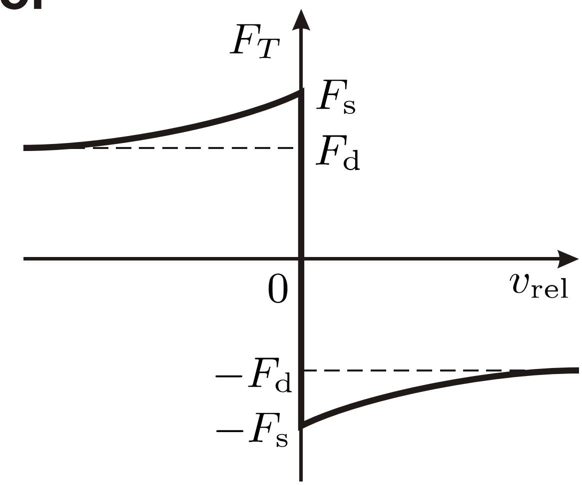
The $n-1$ eigenvalues of

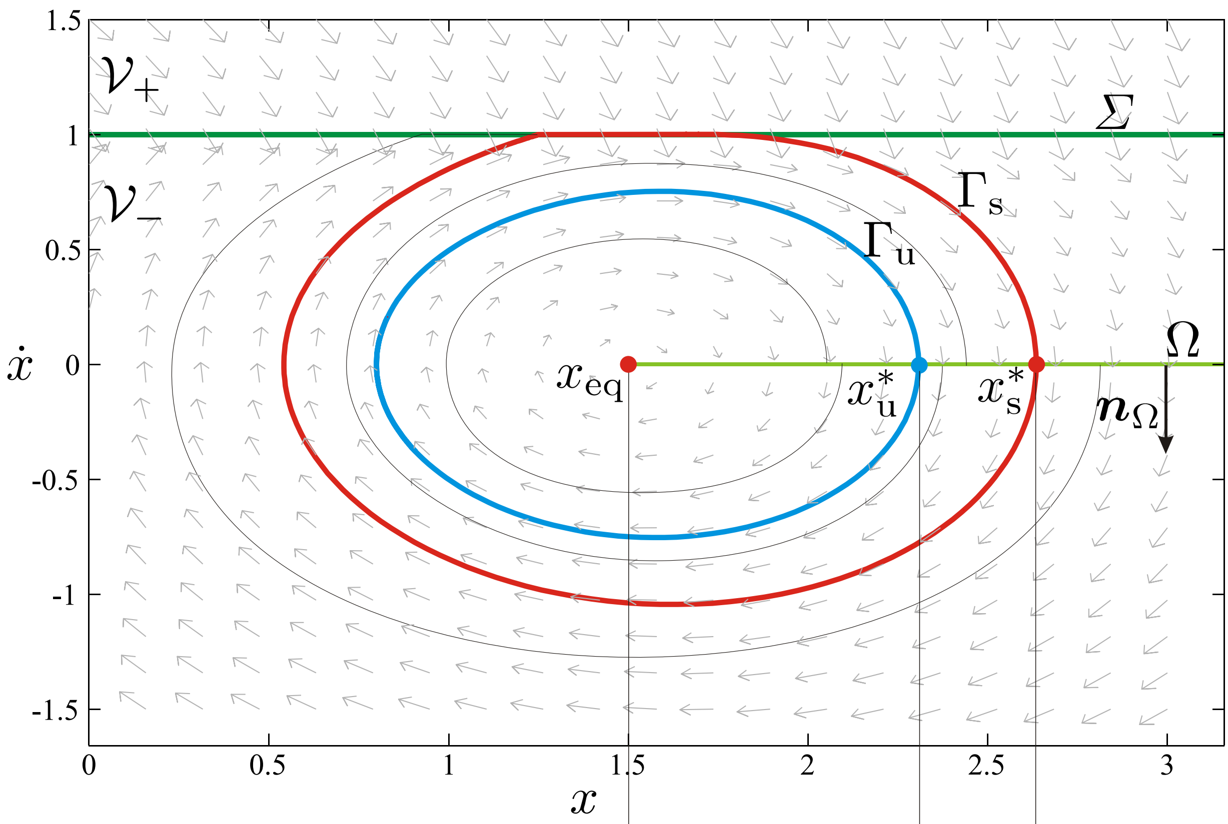
$$\boldsymbol{A} = \left.\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{y}}\right|_{\boldsymbol{y}^*}$$

are identical to the $n-1$ Floquet multipliers $\lambda_2, \ldots, \lambda_n$

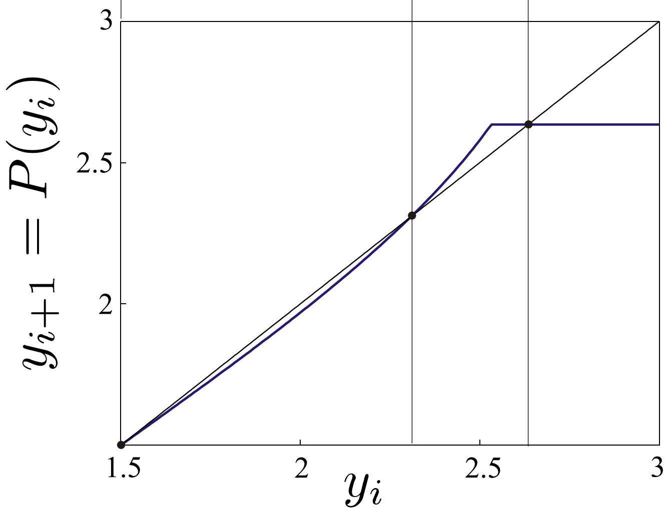
Example: Stick-slip Oscillator

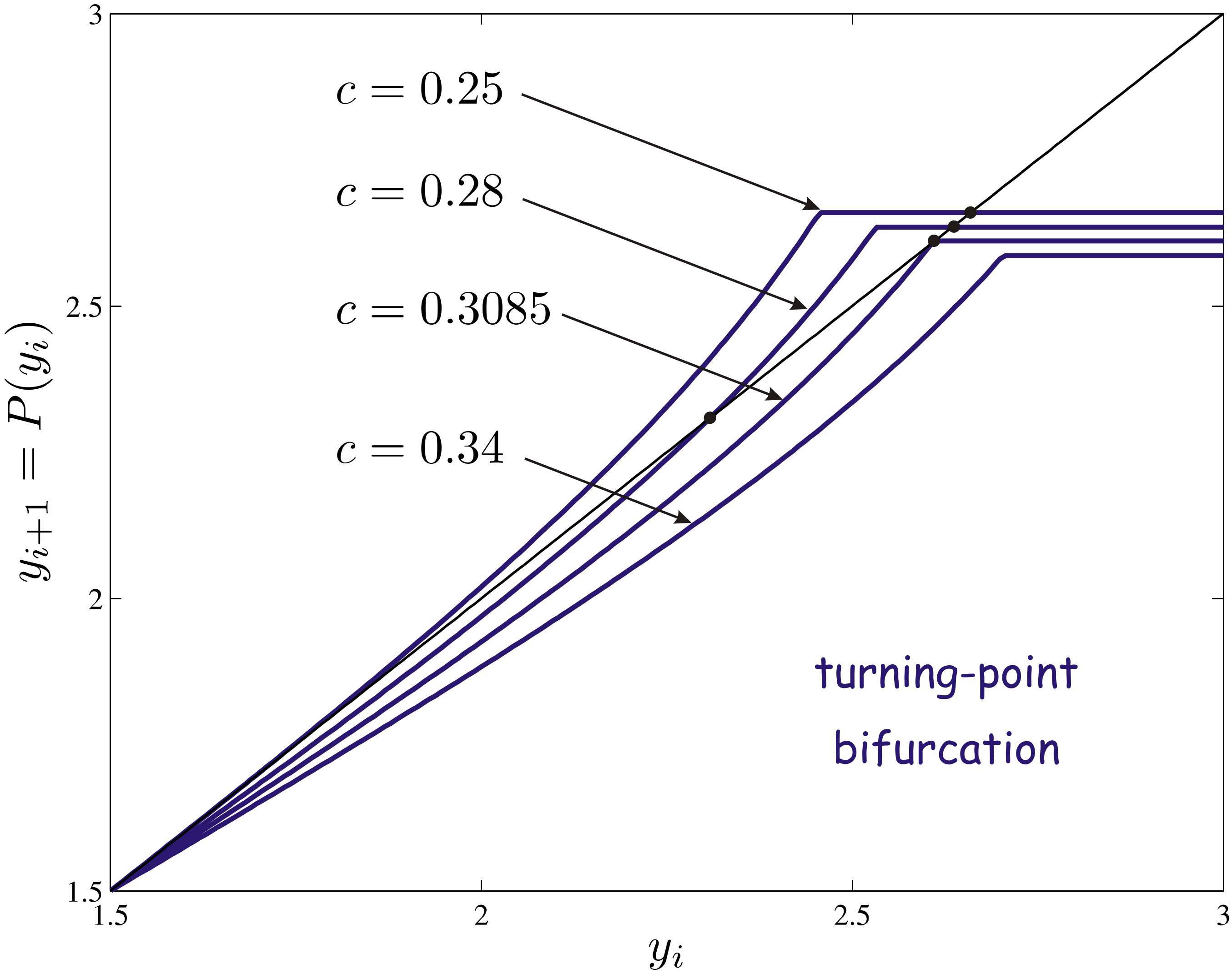






non-smooth continuous map!





discontinuous turning-point bifurcation in the Poincaré map

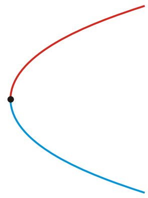
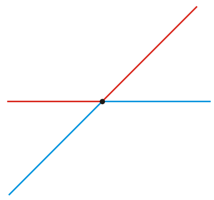
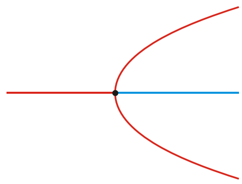
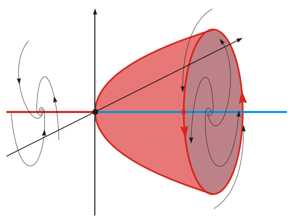
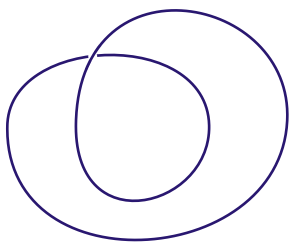
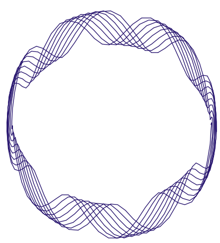
$\Updownarrow$

discontinuous fold bifurcation of the periodic solution

If the eigenvalues of the Poincaré map jump,

then also the Floquet multipliers jump!

7.3 Overview of Bifurcations

Bifurcation in the Poincaré map	**Bifurcation in the continuous-time system**
turning-point bifurcation	saddle-node bifurcation (eq.) or fold bifurcation (per.sol.) 
transcritical bifurcation	transcritical bifurcation (eq.) 
pitchfork bifurcation in the period-1 map	pitchfork bifurcation (eq.)  or Hopf bifurcation (eq.+per.sol.) 
flip bifurcation in the period-1 map = pitchfork bifurcation in the period-2 map	period-doubling bifurcation (per.sol.) 
Naimark-Sacker bifurcation	Naimark-Sacker bifurcation (per.sol.+quasi-per.sol.) 

can all be discontinuous!

7.4 Saltation Matrix for Filippov Systems

Filippov system:

$$\dot{\boldsymbol{x}}(t) \in \boldsymbol{F}(t, \boldsymbol{x}(t)) = \begin{cases} \boldsymbol{f}_-(t, \boldsymbol{x}(t)), & \boldsymbol{x} \in \mathcal{V}_- \\ \overline{\text{co}}\{\boldsymbol{f}_-(t, \boldsymbol{x}(t)), \boldsymbol{f}_+(t, \boldsymbol{x}(t))\}, & \boldsymbol{x} \in \Sigma \\ \boldsymbol{f}_+(t, \boldsymbol{x}(t)), & \boldsymbol{x} \in \mathcal{V}_+ \end{cases}$$

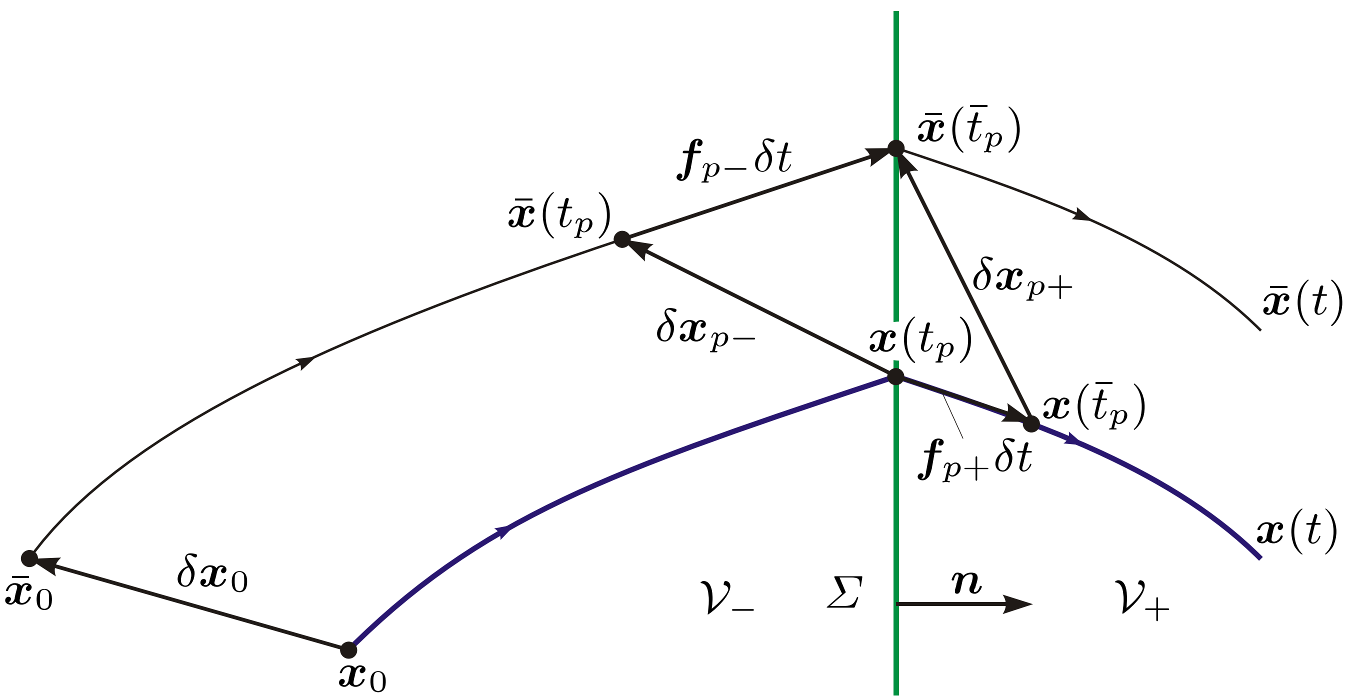
almost everywhere

variational equation: $\dot{\boldsymbol{\Phi}}(t) = \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_p(t)} \boldsymbol{\Phi}(t) \qquad \boldsymbol{\Phi}(t_0) = \boldsymbol{I}$

Jacobian: $\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}_p(t)}$ does **not** exist everywhere!

The fundamental solution matrix **jumps** $\boldsymbol{\Phi}(t_{p+}) = \boldsymbol{S}\boldsymbol{\Phi}(t_{p-})$

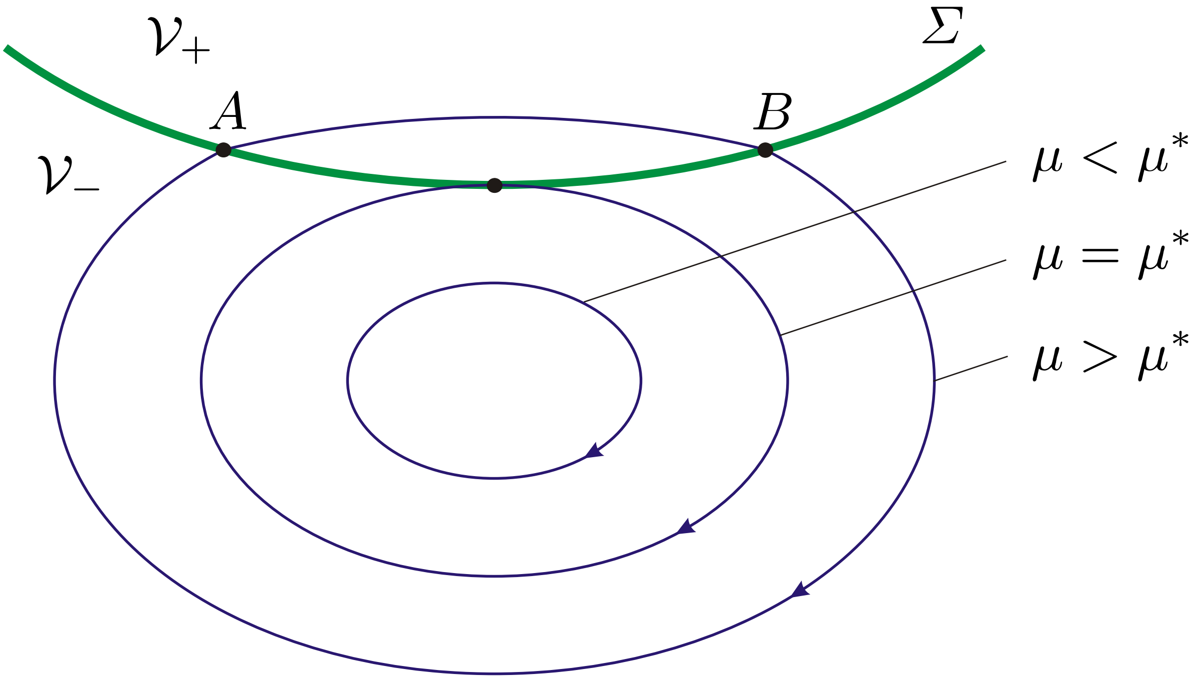
with the saltation matrix $\boldsymbol{S} = \boldsymbol{I} + \frac{(\boldsymbol{f}_{p+} - \boldsymbol{f}_{p-})\boldsymbol{n}^{\mathrm{T}}}{\boldsymbol{n}^{\mathrm{T}}\boldsymbol{f}_{p-}}$.



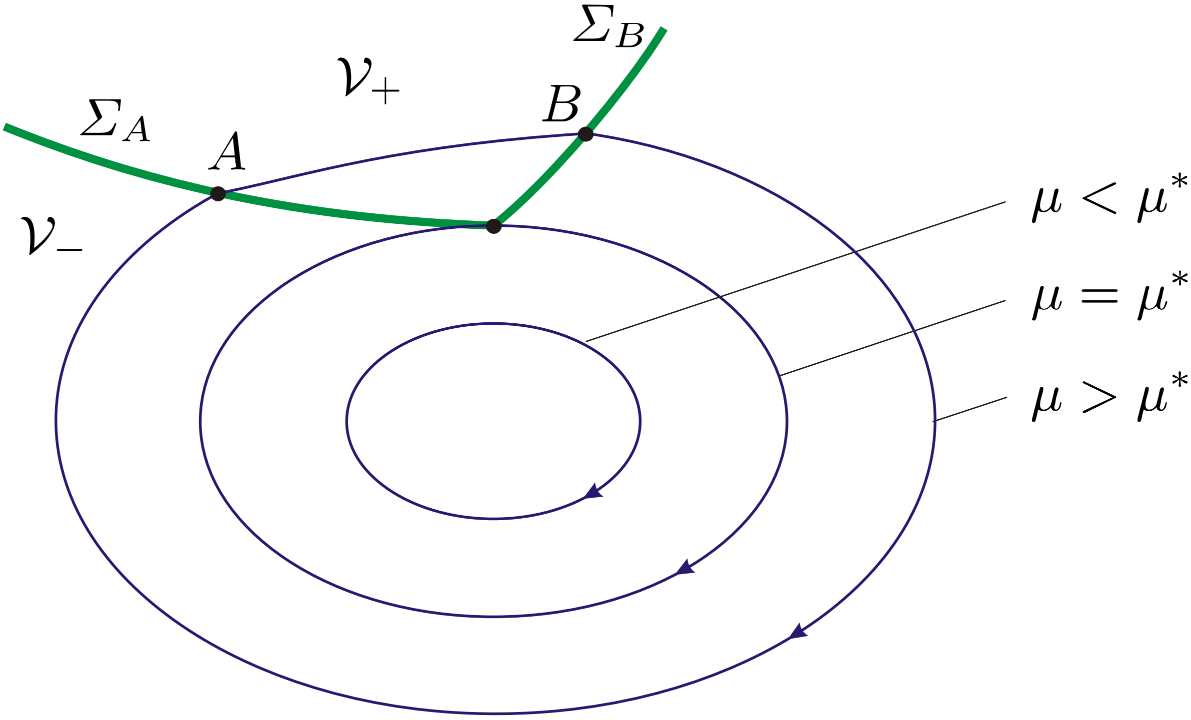
$$\boldsymbol{S}_{BA} = \boldsymbol{S}_B \boldsymbol{S}_A$$

if A and B are infinitely close to each other:

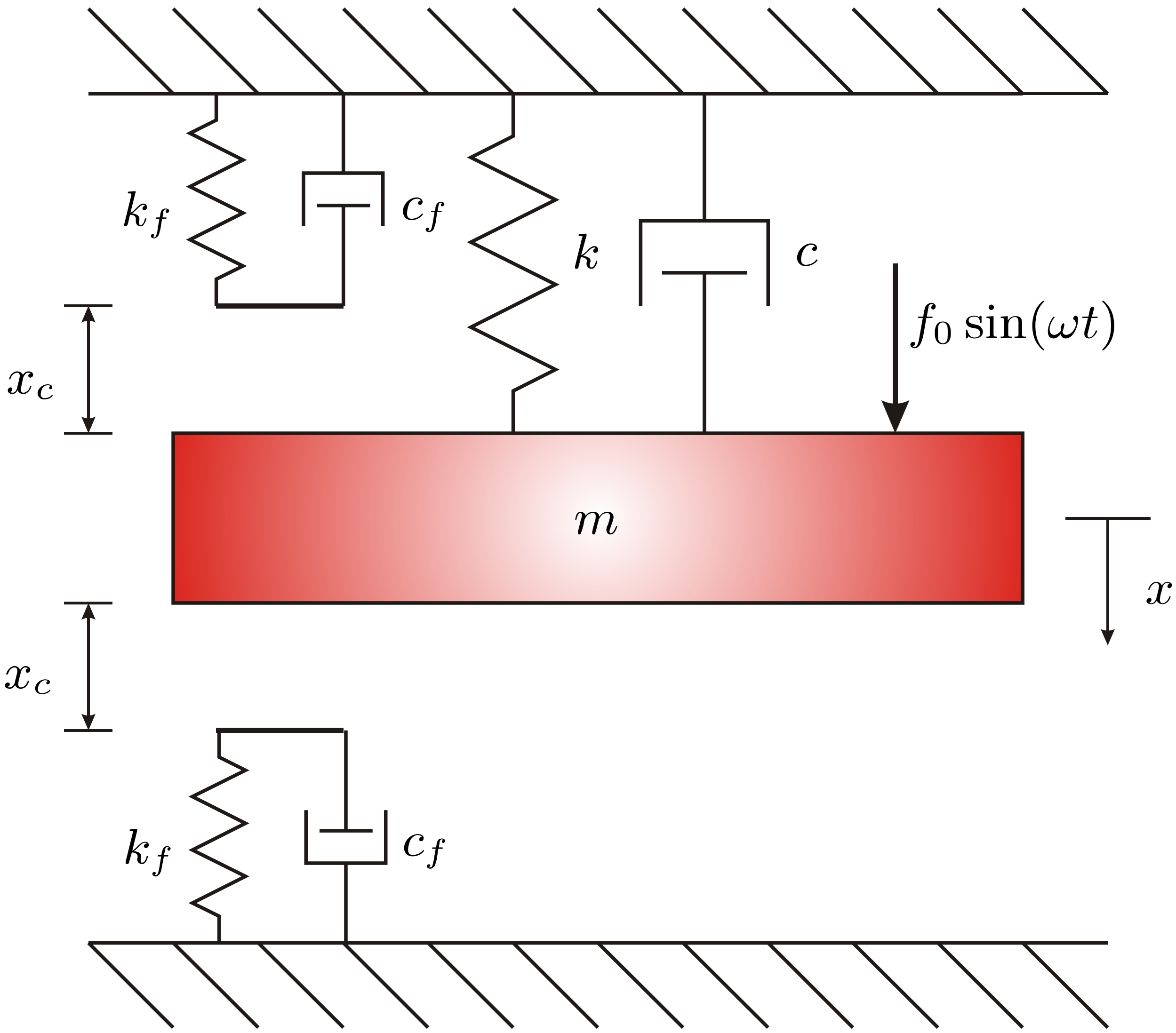
$\boldsymbol{S}_A = \boldsymbol{S}_B^{-1} \implies \boldsymbol{S}_{BA} = \boldsymbol{I} \implies$ no jump in $\boldsymbol{\Phi}_T$



$\boldsymbol{S}_A \neq \boldsymbol{S}_B^{-1} \implies$ jump in $\boldsymbol{\Phi}_T$



7.4 Trilinear Spring System

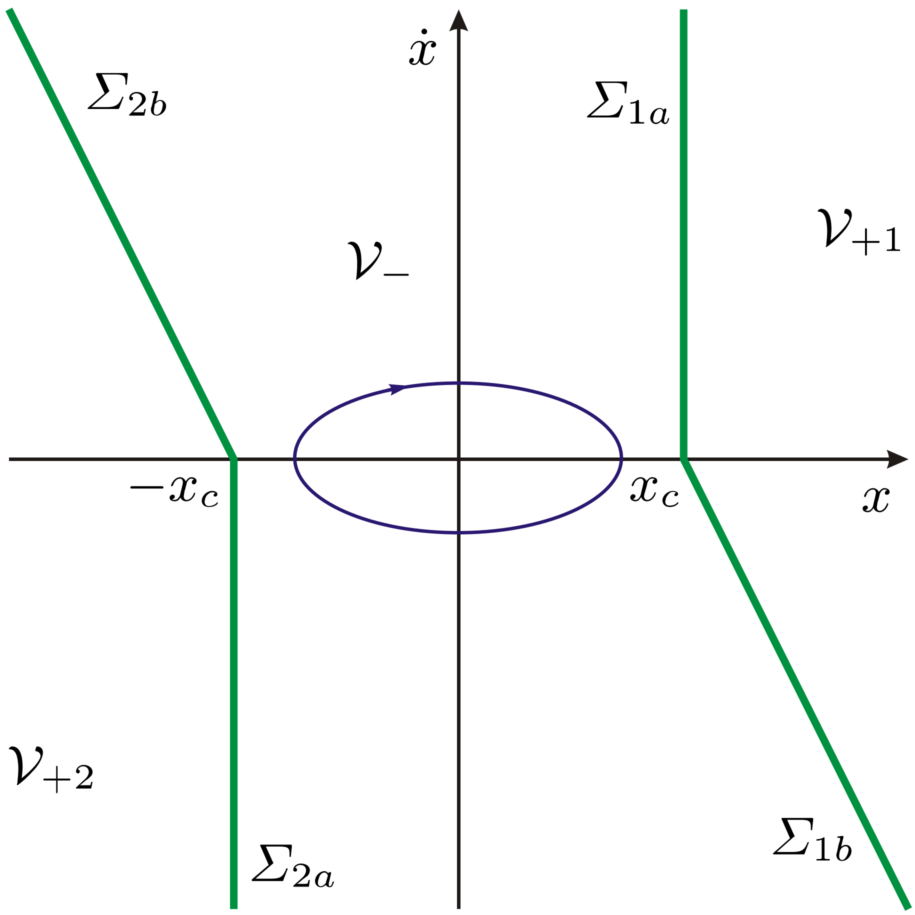


supports are massless

Vector field jumps if contact is made

$\Downarrow$

Filippov system



$$h_{1a}(x, \dot{x}) = x - x_c$$

$$h_{1b}(x, \dot{x}) = k_f(x - x_c) + c_f \dot{x}$$

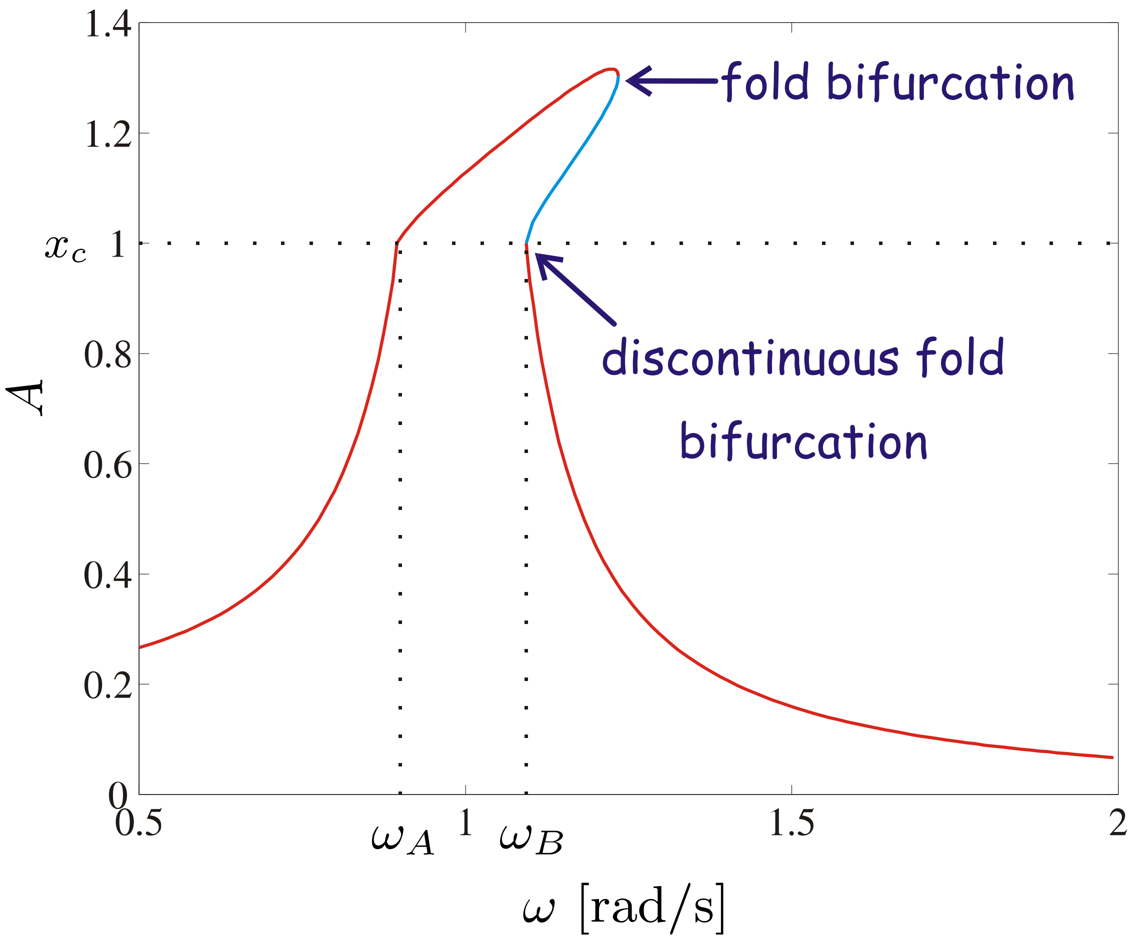
$$h_{2a}(x, \dot{x}) = x + x_c$$

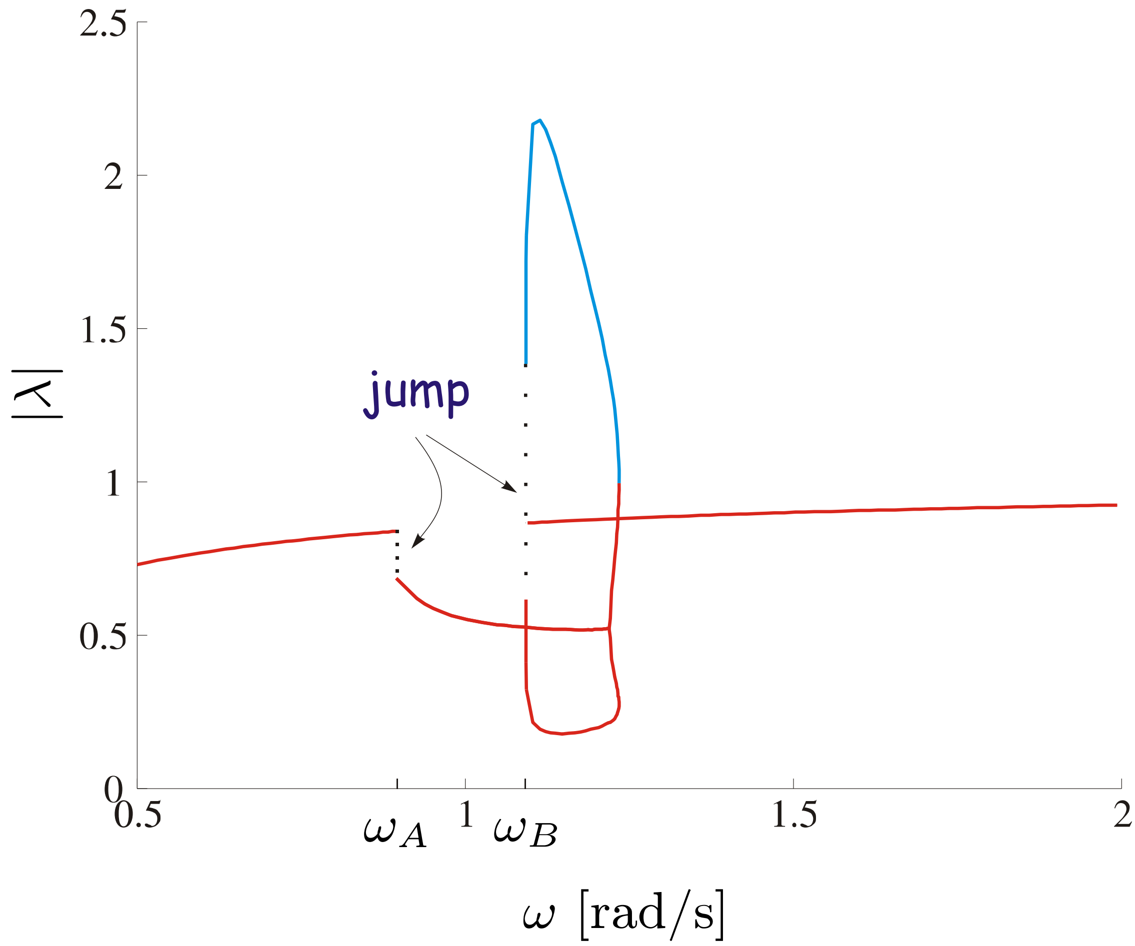
$$h_{2b}(x, \dot{x}) = k_f(x + x_c) + c_f \dot{x}$$

$$\boldsymbol{S}_{1a} = \boldsymbol{S}_{2a} = \begin{bmatrix} 1 & 0 \\ -\frac{c_f}{m} & 1 \end{bmatrix} \qquad \boldsymbol{S}_{1b} = \boldsymbol{S}_{2b} = \boldsymbol{I}$$

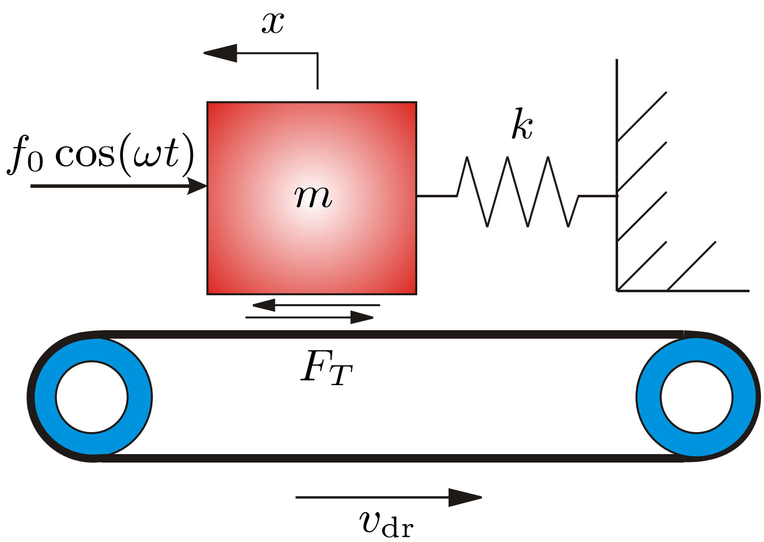
saltation matrices on each side of the corner points are not each others inverse:

$$\boldsymbol{S}_{1a} \neq \boldsymbol{S}_{1b}^{-1} \qquad \boldsymbol{S}_{2a} \neq \boldsymbol{S}_{2b}^{-1}$$


fold bifurcation
discontinuous fold bifurcation
A
x_c
ω_A
ω_B
ω [rad/s]

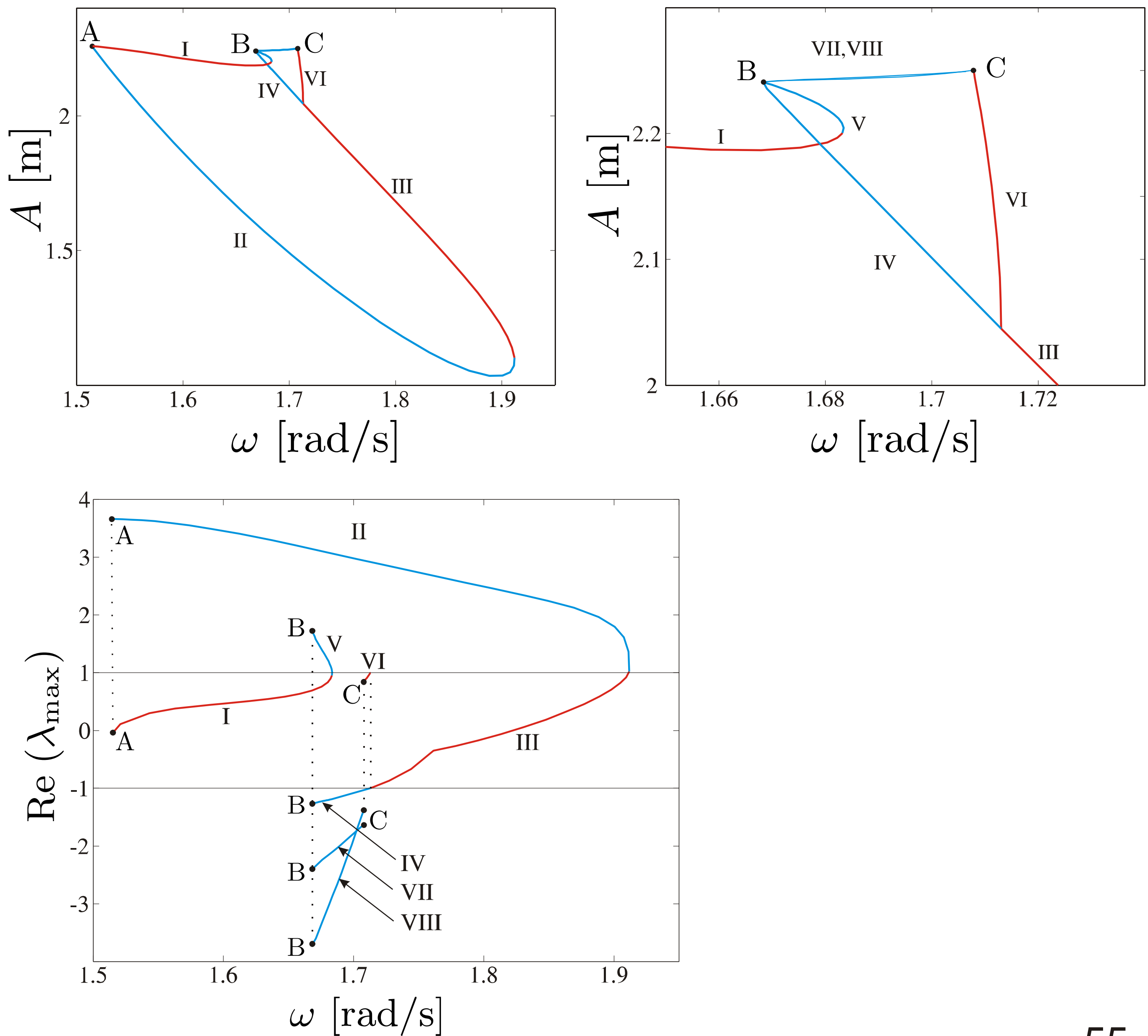

jump
$|\lambda|$
ω_A
ω_B
ω [rad/s]

7.5 Stick-slip System with External Excitation



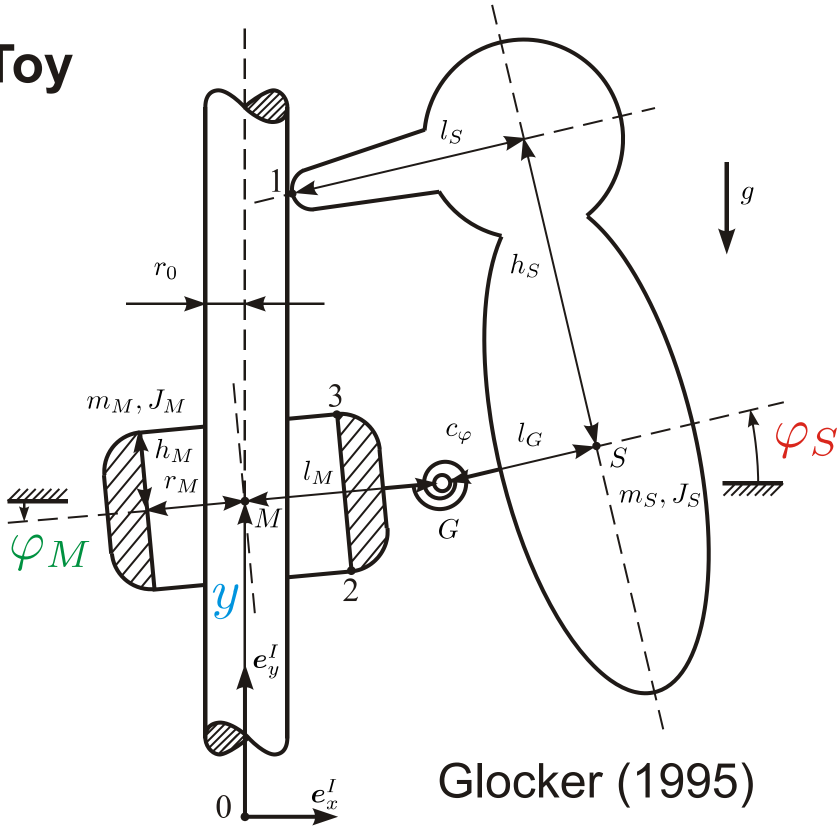
non-autonomous

Filippov system

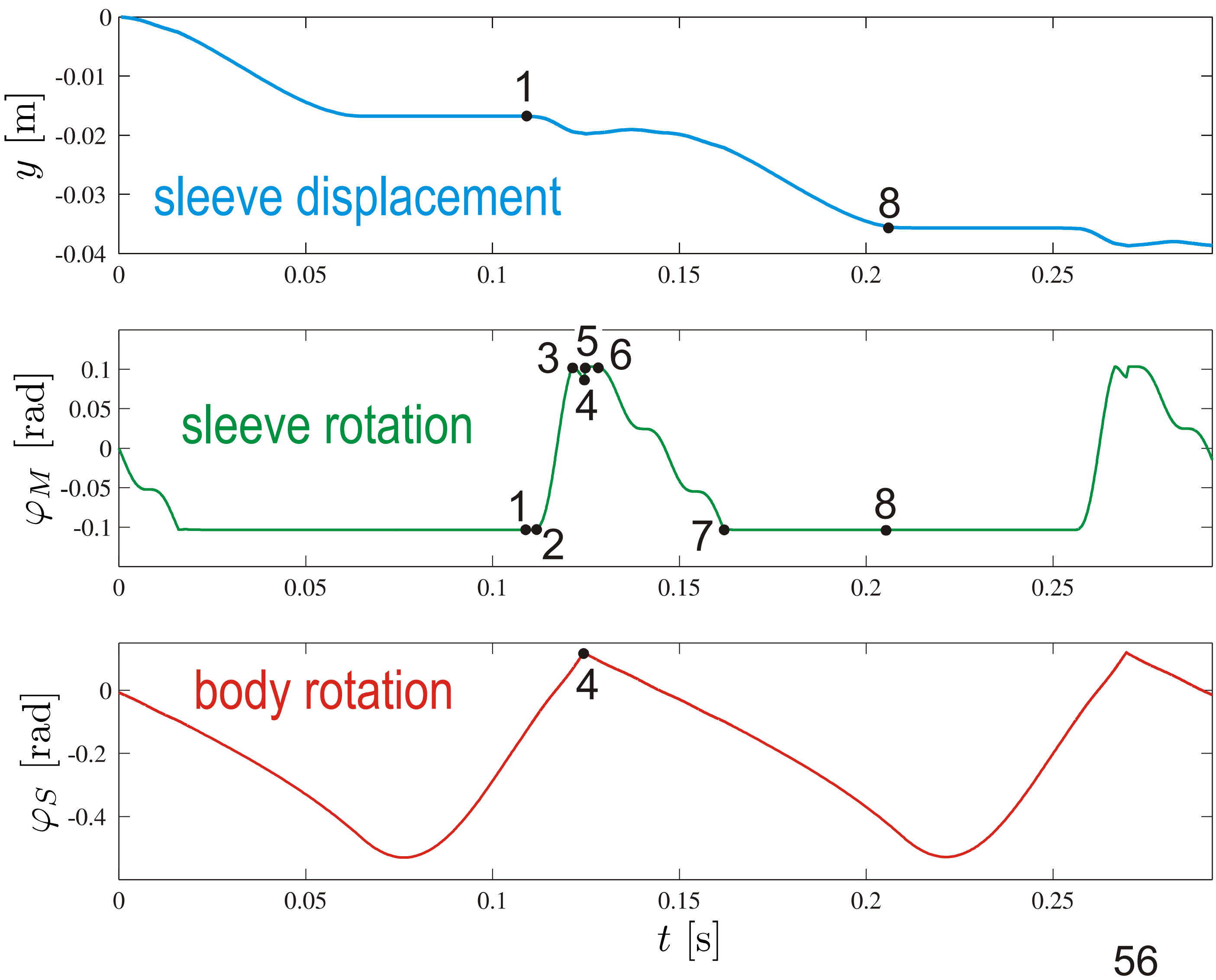


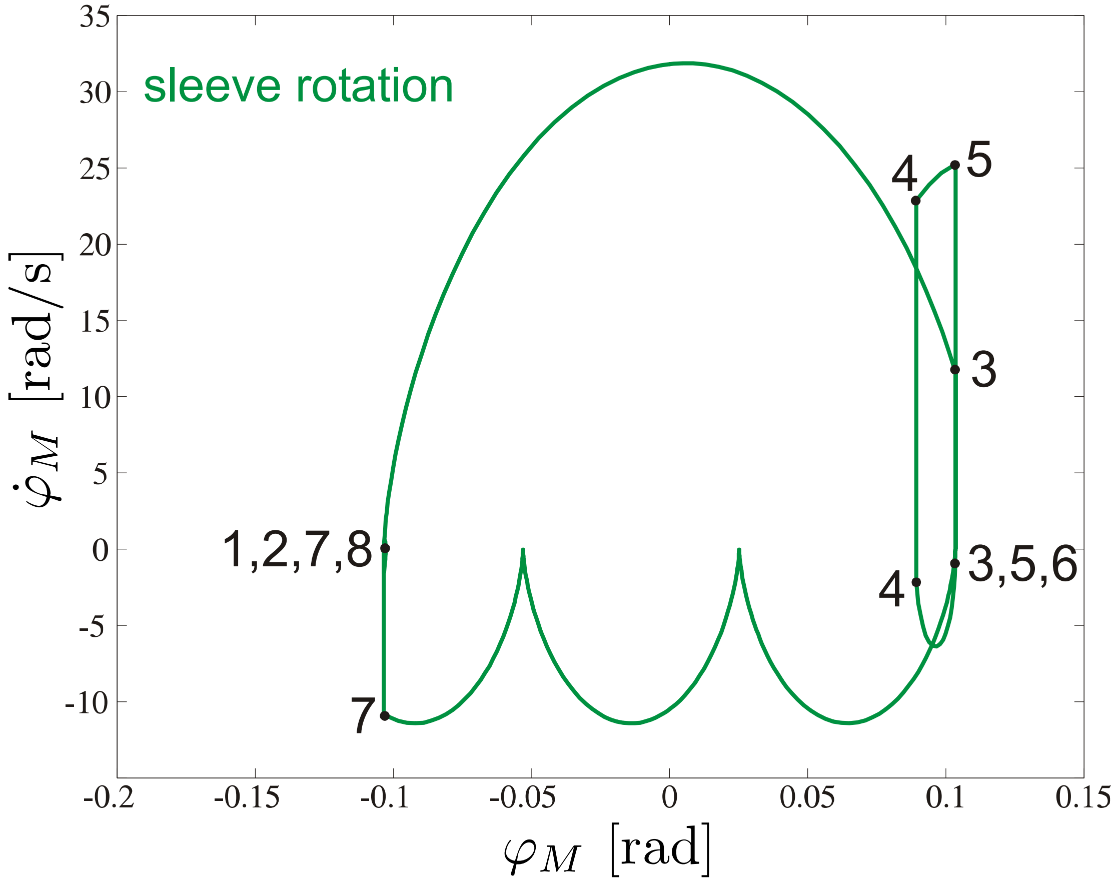
7.6 The Woodpecker Toy

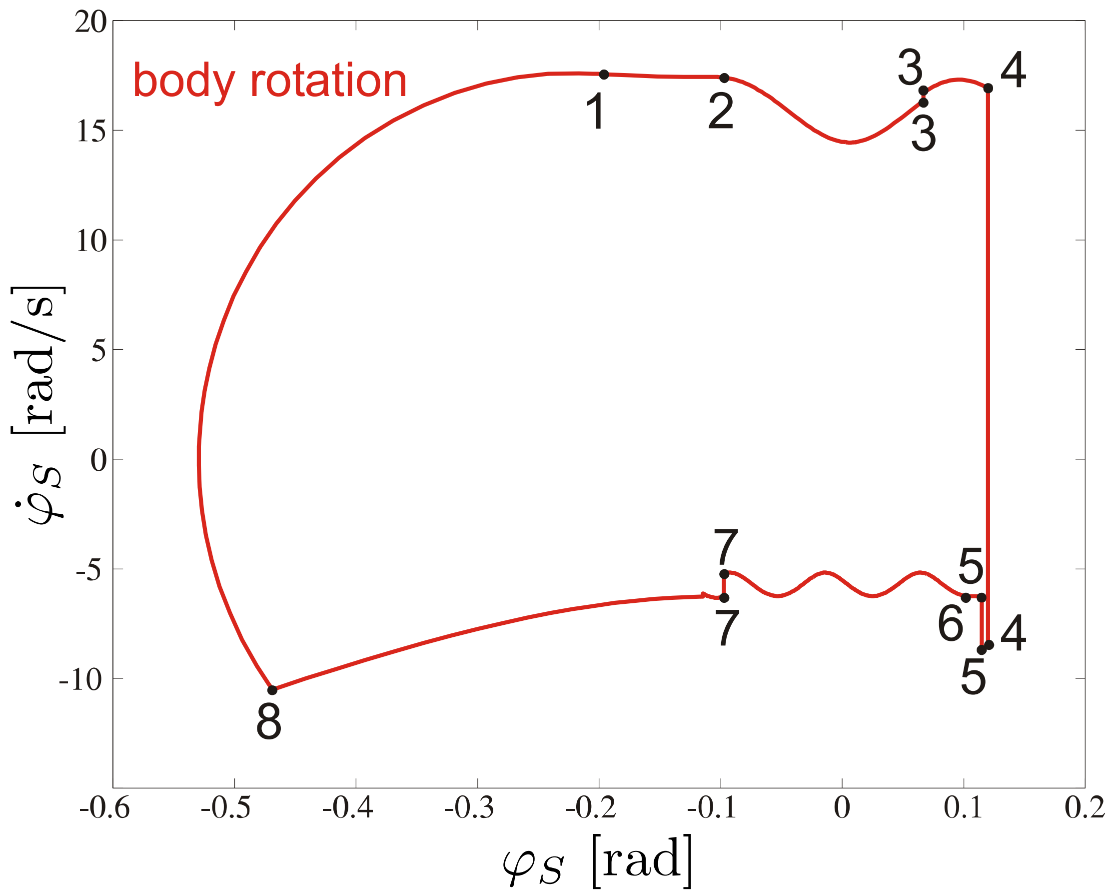
Pfeiffer (1984)
Glocker (1995,2005)
Leine (2003)



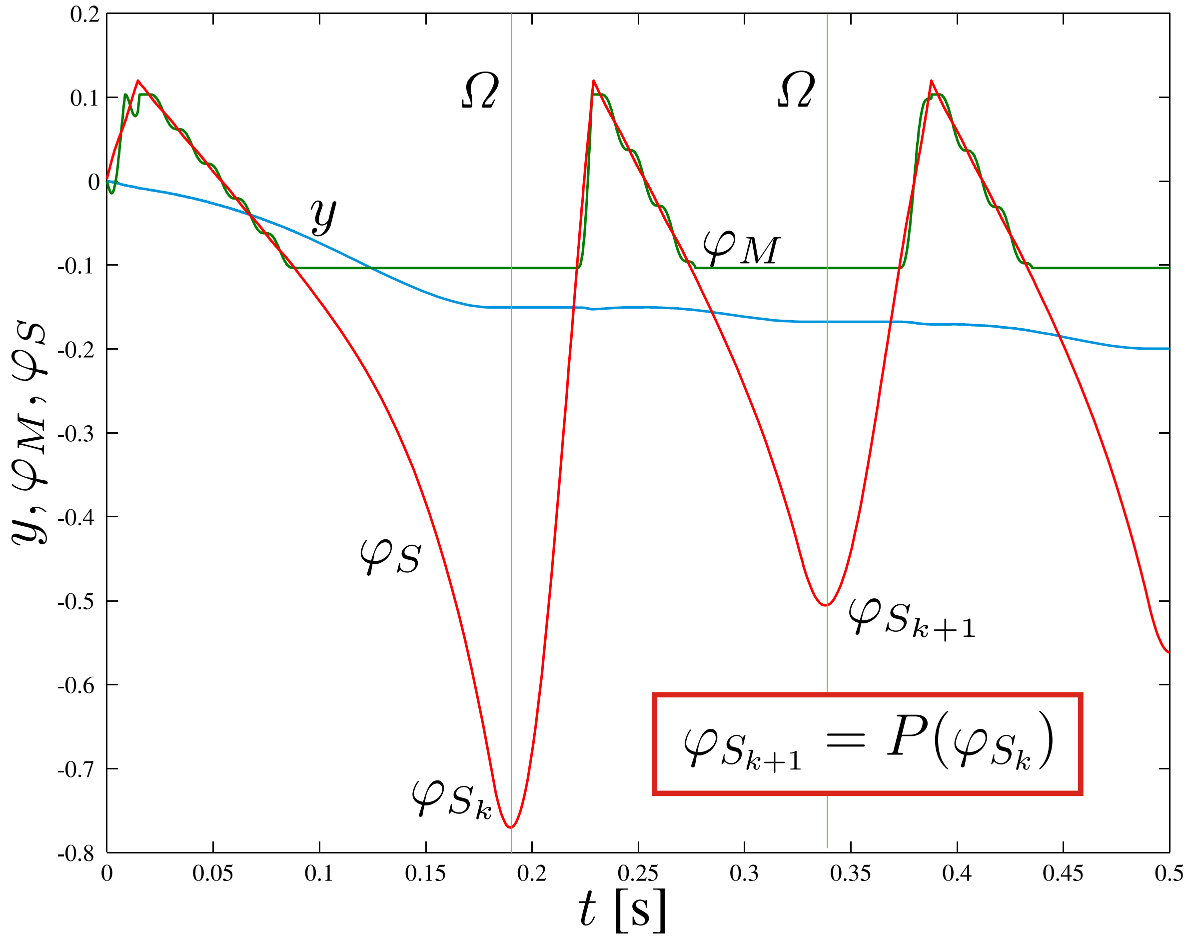
Glocker (1995)

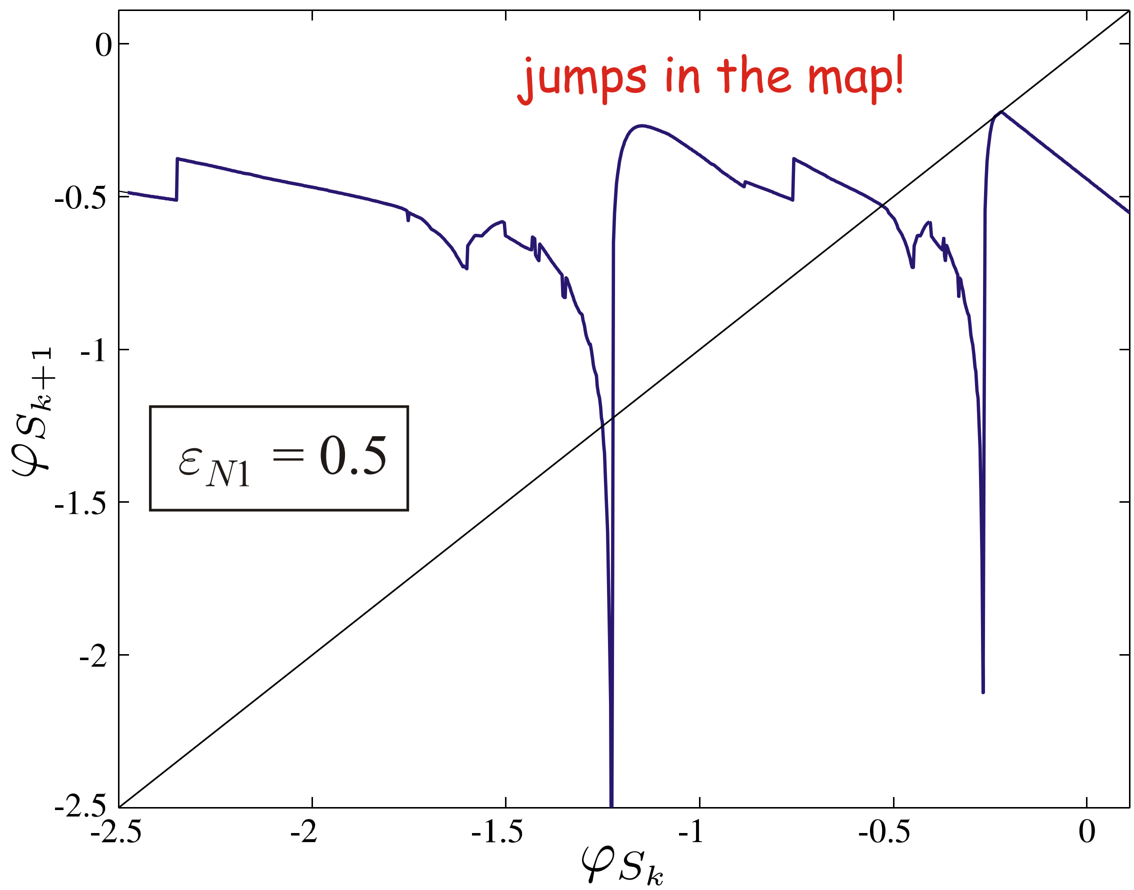



sleeve rotation
1,2,7,8
3,5,6
$\dot{\varphi}_M$ [rad/s]
φ_M [rad]

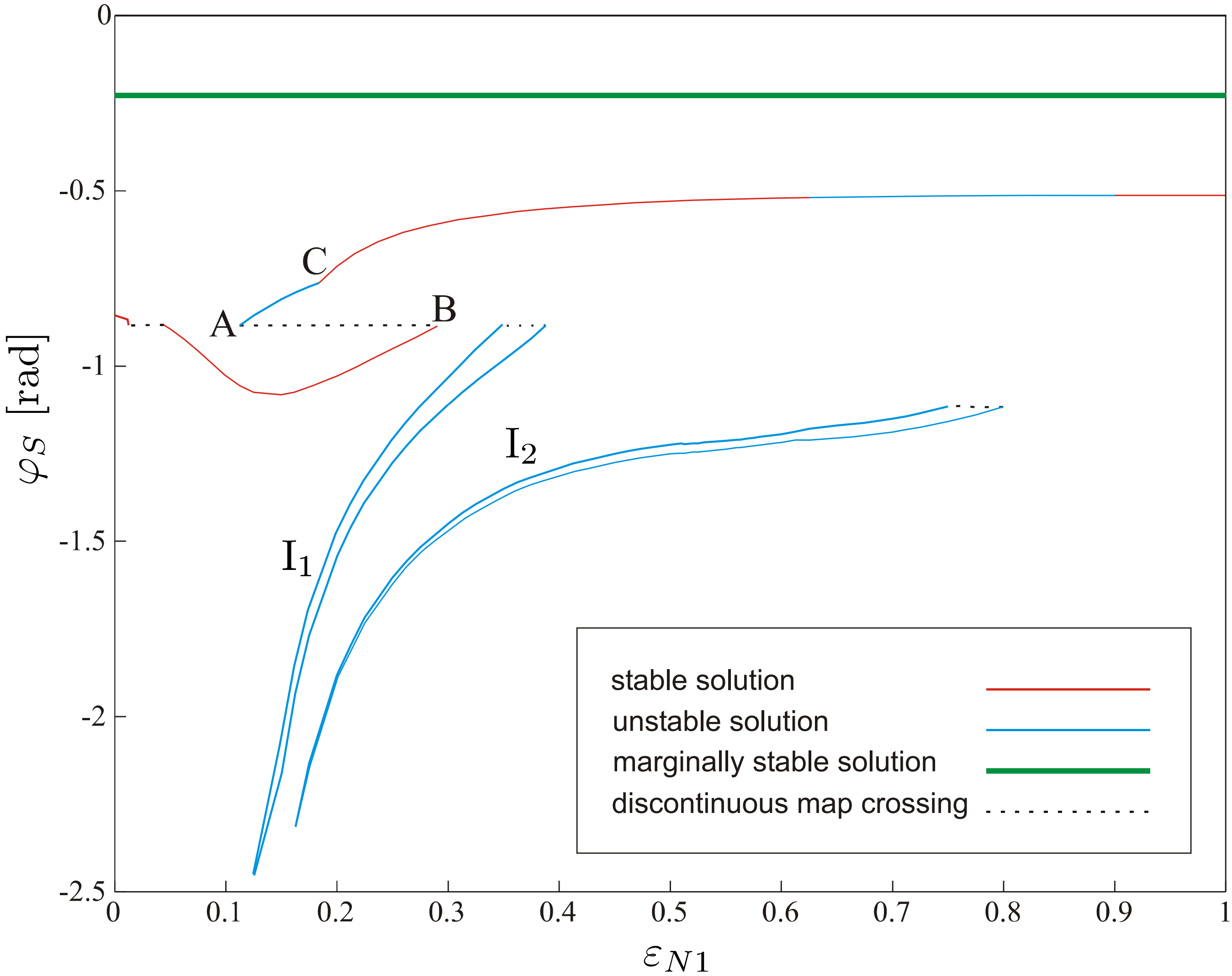

body rotation
$\dot{\varphi}_S$ [rad/s]
φ_S [rad]

The dynamics is described by a one-dimensional map

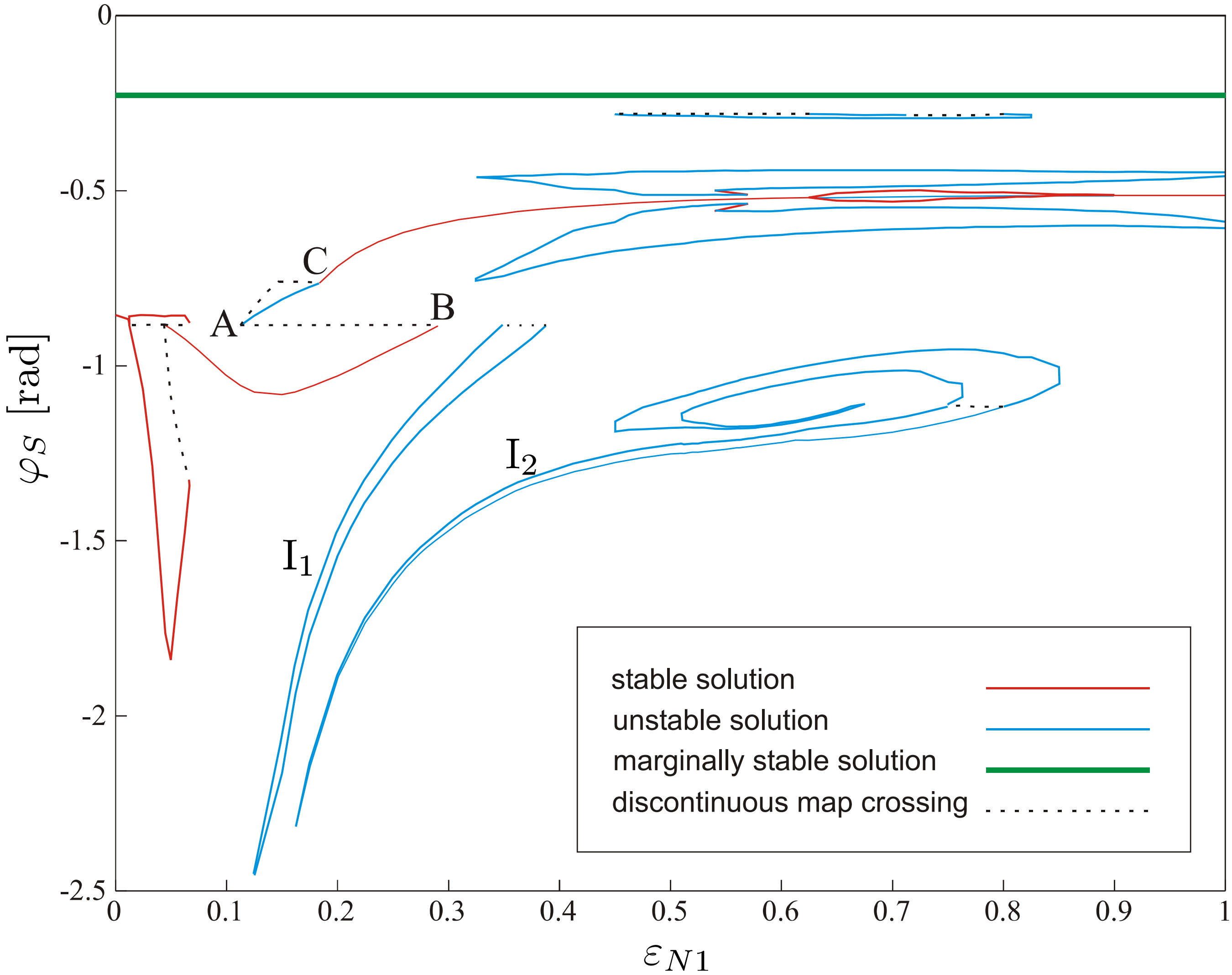


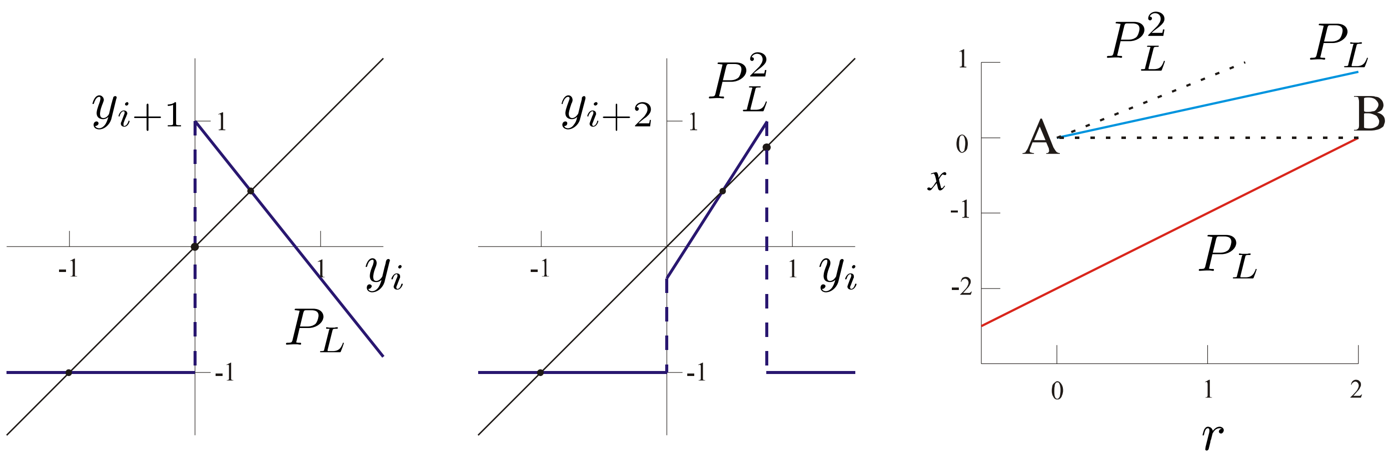


Map of period-1 solutions



Map of period-1 and -2 solutions

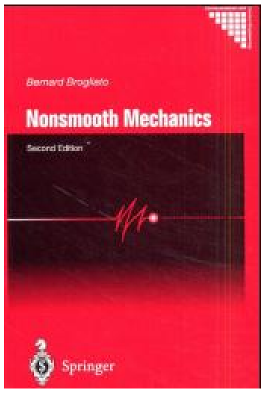




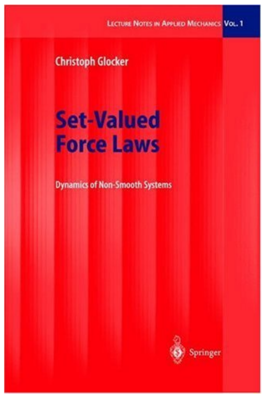
$$P_L(y) = \begin{cases} -2 + r & y \le 0 \\ -\frac{5}{4}y + r & y > 0 \end{cases}$$

8 Literature

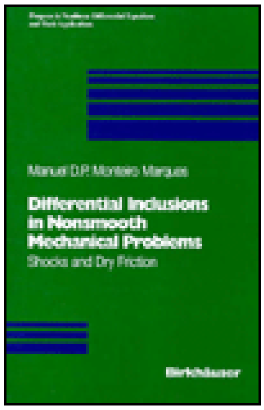
Non-smooth Systems



Brogliato, B. *Nonsmooth Mechanics*, 2ed., Springer, London, 1999.

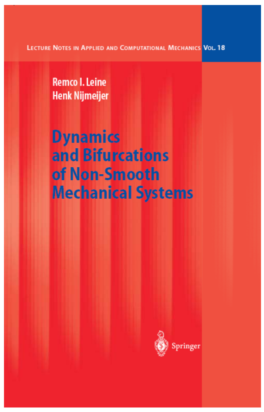


Glocker, Ch. *Set-valued Force Laws, Dynamics of Non-smooth Systems*, Lecture Notes in Applied Mechanics Vol.1, Springer, Berlin, 2001.

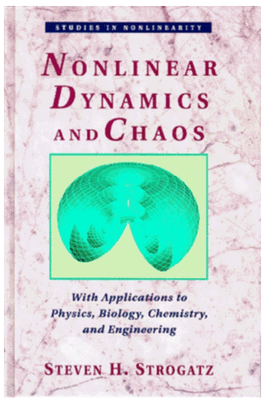


Monteiro Marques, M.D.P. *Differential Inclusions in Nonsmooth Mechanical Problems. Shocks and Dry Friction*, Birkhäuser, Basel, 1993.

Nonlinear Dynamics



Leine, R.I. & Nijmeijer, H. *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*, Lecture Notes in Applied and Computational Mechanics Vol.18, Springer, Berlin, 2004.



Strogatz, S. *Nonlinear Dynamics and Chaos*, Studies in Nonlinearity, Addison-Wesley, Reading, 1994.